Math 6528 Notes
Real Analysis

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MEASURES

Notation: If E is any set, then the characteristic function of E is given by

$$\chi_E(x) = \begin{cases} 1, x \in E \\ 0, x \notin E \end{cases}$$

Step Functions: S(x) is a *step function* if S(x) is piecewise constant on intervals. If one partitions an interval [a,b] into sub-intervals, a step function can be written of the form

$$s(x) = \sum_k s_k \chi_k(x),$$

where s_k is a value of the function on interval k.

Further, the integral of this step function is given by $\sum_k s_k \Delta x_k$, where Δx_k = length of interval k.

Result: To determine the Riemann integral for a general function, partition the interval [a,b]. To obtain an *upper sum* create the step function $S_u(x)$ such that on each subinterval $S_u(x)$ = max f(x), for all x's in the subinterval. Similarly, one can create *lower sums*.

Riemann Integral: A general function f:[a,b]→**R** is *Riemann integrable* if the infinum of all upper sums equals the supremum of all lower sums.

To integrate a general function f(x), take a sequence of step functions on a finer and finer partition of [a,b] and define the Riemann integral of f(x) to be the limit of the integrals of the step functions.

Questions: Is the function f integrable over [0,1] when f is given by:

- f(x) = x; YES (Easy function using FTOC)
- f(x) = sin(x); YES (Transcendental using FTOC)
- f(x) = (x - 0.5) / |x - 0.5|; YES (Note, this is discontinuous.)
- $f(x) = \chi_{\{1/n \mid n=1,2,3,...\}}(x)$; YES
- $f(x) = \chi_Q(x)$, where Q=rationals; NO (There are functions which are not Riemann integrable.)

Problem: There exist Riemann integrable functions $f_n(x)$ such that $f_n \to \chi_Q$. Indeed, take the set

$$E_n = \{p/q \mid \text{the fraction is reduced and } q \le n,\ 0 \le p \le q\}.$$

Note, as n gets larger, E_n gets closer to the set of rationals Q. Also, notice the sequence of functions

$$f_1 = \chi_{\{0,1\}}(x), \quad f_2 = \chi_{\{0,1/2,1\}}(x), \quad f_3 = \chi_{\{0,1/3,1/2,2/3,1\}}(x), \ ...$$

are all Riemann integrable on the interval [0,1] but they approach $\chi_Q(x)$ which isn't Riemann integrable on the interval [0,1]. This problem arises because the Riemann integral is not "complete". Thus, the need for the a new way of determining whether something is integrable…enter the Lebesgue integral.

Generalization: Instead of using interval partitions of [a,b] and the resulting step functions to define the integral (like Riemann), use a collection of disjoint sets E_k (not necessarily intervals) such that the union of the E_k = [a,b] and create a *simple function*

$$s(x) = \sum_k s_k \chi_{E_k}(x).$$

Then, the integral of this simple function is given by

$$\sum_k s_k m(E_k)$$

where the measure of the set E_k = $m(E_k)$, generalizes length.

Defn: For given sets E and F:

- *The complement* of E is the set $E^c = \{x \mid x \notin E\}$
- *The set difference* $E - F = \{ x \mid x \in E \text{ and } x \notin F \}$
- Two sets are *disjoint* if their intersection is the empty set.
- The *superior limit* of $\{E_n\}$, denoted lim sup E_n, is the set consisting of those points which belong to infinitely many of the E_n.
- The inferior limit of $\{E_n\}$, denoted lim inf E_n, is the set consisting of those points which belong to all but a finite number of the E_n.
- If E^* = lim sup E_n = lim inf E_n, then we say the sequence $\{E_n\}$ has a *limit* E^*.

Defn: (Friedman) Ring, algebra, σ-ring and σ-algebra ... see defns 1.1.1 and 1.1.2.

Note: The definitions above require closure with respect to set union and set difference. It is easy to show that in a ring and σ-ring, we also get closure with respect to set intersection by considering $E \cap F = E - (E-F)$. Also, in an algebra and σ-algebra, we get closure with respect to complements by considering X-E.

HOMEWORK: (Friedman) page 3, #3 (assuming σ-algebra), 4

Borel's Conditions: Properties that a good measure should have.

- B0: m([a,b]) = length([a,b]) = b-a.
- B1: $m(E) \geq 0$, for any set E
- B2: $m(\cup E_k) = \Sigma\, m(E_k)$, for mutually disjoint sets E_k.
- B3: m(F - E) = m(F) - m(E), where E is contained in F
- B4: m(E) > 0 implies E is uncountable

Defn: A given set function m with domain a ring **R** is said to be:

- *additive* provided $m(E \cup F) = m(E) + m(F)$, where $E \in \mathbf{R}$, $F \in \mathbf{R}$ and $E \cap F = \phi$.
- *finitely additive* if $m(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m(E_k)$, where all of the sets are mutually disjoint
- *completely additive* if $m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k)$, where all of the sets are mutually disjoint

Defn 1.2.1: A *measure* is an extended real-valued set function m having the following properties:

- The domain **A** of m is a σ-algebra.
- m is nonnegative on **A**
- m is completely additive on **A** (see fomula 1.2.1, page 4)
- m(ϕ) = 0

Result: A measure is finitely additive.

Pf: Since a measure is completely additive, it must be finitely additive by taking all the sets E_m to be empty, m > n.

Defn: If X is the entire space under consideration and m(X) < ∞, then we say the set function m is a *finite measure*. If X can be written as the infinite union of sets E_n such that for all n, m(E_n) < ∞, then the measure is a σ-finite measure.

Theorem 1.2.1. Let m be a measure with domain **A**. Then:

(Monotonicity – HW #1, pg 55) If E$\in$**A** and F$\in$**A** with E$\subseteq$F, then m(E) $\le$ m(F).

Pf: Write F = E U (F-E), which is a disjoint union.
Since a measure is additive, then m(F) = m(E) + m(F-E).
But a measure is also nonnegative and so m(F-E) $\ge$ 0. Hence, the result follows.

(Differences) If E$\in$**A** and F$\in$**A** with E$\subseteq$F and m(F)< ∞ , then m(F-E) = m(F) - m(E).

Pf: Use formula above.

(Continuity) If {E_n} is a monotone-increasing sequence in **A**, then lim m(E_n) = m(lim E_n).

Pf: Problem 1.1.3 (Friedman) implies that lim E_n is in the domain **A**.
If $E_0 = \phi$, notice $E_n = (E_n - E_{n-1}) \cup (E_{n-1} - E_{n-2}) \cup \ldots \cup (E_2 - E_1) \cup (E_1 - E_0) = \cup (E_k - E_{k-1})$.
As n→∞, this becomes an infinite union of mutually disjoint sets.

Thus, $m(\lim E_n) = m(\cup (E_k - E_{k-1})) = \sum_{k=1}^{\infty} m(E_k - E_{k-1})$, by completely additivity

$= \lim_{n\to\infty} \sum_{k=1}^{n} m(E_k - E_{k-1})$, going to the definition of an infinite summation

$= \lim_{n\to\infty} m(\bigcup_{k=1}^{n}(E_k - E_{k-1}))$, using finitely additive,

= lim m(E_n), noting the union collapses.

(Continuity Revisited) The limits can be interchanged also provided the sequence is monotone-decreasing and the measure of one of the sets E_M is finite.

Pf: Notice that the sequence $E_M - E_k$ is now monotone-increasing. Apply (iii) and (ii).

Theorem 1.2.2: (HW #2, pg 55)
Let m be a measure with domain **A**. Then, for the infinite collection {E_n}, n=1..., of sets of **A**,

$$m(\cup E_n) \le \Sigma\, m(E_n)$$

Pf: Create the mutually disjoint, monotone sequence of sets F_n using

$$F_1 = E_1 \text{ and } F_n = E_n - [E_1 \cup E_2 \cup \ldots \cup E_{n-1}],$$

a mutually disjoint collection of sets. Hence,

$$m(\cup E_n) = m(\cup F_n) = \Sigma\, m(F_n) \leq \Sigma\, m(E_n).$$

HOMEWORK: (Friedman) page 7, #1, 3; (Royden) page 54, #3, 4

Defn: Given a set $A \subseteq \Re$ and an interval measure λ defined on open intervals, the outer measure is

$$m^*(A) = \inf_{A \subset \cup I_n} \sum \lambda(I_n).$$

Results: Given any outer measure m*:

- $m^*(\phi) = 0$
- $A \subseteq B$ implies $m^*(A) \leq m^*(B)$
- $m^*(\{x\}) = 0$

Proposition 1: The outer measure of an interval is its length.
Pf:
<u>Case 1</u>: A=[a,b].
Easily, for ε>0, $[a,b] \subseteq (a - \varepsilon, b + \varepsilon)$. Therefore, $m^*([a,b]) < b - a + 2\varepsilon$.
Since this is true for any such ε, then

$$b - a = \inf \lambda(a - \varepsilon, b + \varepsilon).$$

Thus $m^*([a,b]) \leq b - a$.
If $\{E_k\}$ is another countable collection of open intervals covering [a,b], then easily $b - a \leq \Sigma\, \lambda(E_k)$.
By the Heine-Borel theorem, this countable cover must contain a finite subcover of [a,b].
WOLOG, consider the subinterval which contains a and denote it E_1 so that

$a \in E_1 = (a_1,b_1)$

If $b_1 \leq b$, continue enumerating from this finite subset of intervals E_2, E_3, …

$b_1 \in E_2 = (a_2,b_2)$,
$b_2 \in E_3 = (a_3,b_3)$, etc.

Since the finite subcover must eventually cover b, this process will end with

$b \in E_n = (a_n,b_n)$.

Therefore,

$$\Sigma\, \lambda(E_n) > \Sigma\, \lambda((a_k,b_k)) = \Sigma\, (b_k - a_k) > b_n - a_1 > b - a$$

and so $b - a < \Sigma\, \lambda(E_n)$. Hence, the infinum can be no smaller than b-a.

<u>Case 2</u>: A = a finite collection of disjoint intervals (including open or closed pieces)
For each interval E in A , there is a closed interval $F \subset E$ such that $\lambda(E) < \lambda(F) + \varepsilon$.
Hence,

$$\lambda(E) - \varepsilon < \lambda(F) = m^*(F) \leq m^*(E) \leq m^*(\overline{E}) = \lambda(\overline{E}) = \lambda(E).$$

Since ε was arbitrary, these inequalities must all be equalities and therefore $m^*(E) = \lambda(E)$

By the finite additivity of m*, the result follows.

Case 3: A = an infinite interval.
Then, for any real number ε, there exists a closed interval $F \subset E$ such that λ(F) = ε.
So,

$$m^*(E) \geq m^*(F) = \varepsilon.$$

This time, let ε become arbitrarily large to yield m*(E) = ∞ = λ(E).

Proposition 2: $m^*(\bigcup E_n) \leq \Sigma\, m^*(E_n)$
Pf: If for one of the sets $m^*(E_n) = \infty$, then the inequality holds trivially.
Therefore, assume the outer measure for each of the sets is finite. By the definition of outer measure, for any ε>0 and for each E_n there is a collection of open intervals $\{I_{n,k}\}$ such that

$$m^*(E_n) + 2^{-n}\varepsilon > \sum \lambda(I_n)$$

with $E_n \subset \bigcup I_{n,k}$. Therefore,

$$m^*(\bigcup E_n) \leq \sum_{n,k} \lambda(I_{n,k}) = \sum_n \sum_k \lambda(I_{n,k}) < \sum_n \left(m^*(E_n) + 2^{-n}\varepsilon\right) = \varepsilon + \sum_n m^*(E_n)$$

Let ε→0 to get the result.

Corollary 3: If A is countable then m*(A) = 0.
Pf: By the result before Proposition 1, the measure of a single point is zero. Since A is countable, then A can be written as a collection of single point sets E_n. By Proposition 2, $m(A) \leq \Sigma\, m^*(E_n) = 0$,

Corollary 4: The set [0,1] is uncountable.
Pf: Using the contrapositive of Corollary 3, m*([0,1]) = 1 > 0 implies A is uncountable.

Defn: Denote by **B** the σ-algebra generated by the class of all open sets of X. The sets of **B** are called *Borel Sets*. **B** is the smallest σ-algebra which contains all the open sets, the smallest σ-algebra which contains all the closed sets and the smallest σ-algebra which contains all the open intervals. (See page 53 – Royden.)

Corollary 5: Given any set A and any ε>0, there is an <u>open</u> set H such that $A \subset H$ and

$$m^*(H) \leq m^*(A) + \varepsilon$$

Further, there is a Borel set G such that $A \subset G$ and m*(A) = m*(G).

HOMEWORK: page 58 #5, 7, 8

Defn: Given an outer measure m*, then the set E is m* measurable if

$$m^*(A) = m^*(A \cap E) + m^*(A-E).$$

Note that m* is subadditive being an outer measure. Hence it is always true that

$$m^*(A) \leq m^*(A \cap E) + m^*(A-E).$$

To show equality, it suffices to verify $m^*(A) \geq m^*(A \cap E) + m^*(A-E)$).

Theorem 1.3.1 (Friedman): Let m* be an outer measure and denote by **A** the class of all m*-measurable sets. Then, **A** is a σ-algebra and the restriction m of m* to **A** is a measure.
Pf:
We will show the following:

- Firstly that **A** is an algebra
- Secondly that m satisfies the measure properties on **A**
- Lastly that **A** is a σ-algebra.

(i) **Lemma 6:** (This will also be used later when discussing completeness)
If $m^*(E) = 0$, then E is measurable.
Pf: For any set A,

$$m^*(A \cap E) + m^*(A - E) \leq m^*(E) + m^*(A) = m^*(A),$$

since m* is monotone and $(A \cap E) \subseteq E$ and $A - E \subset A$

Hence, defn is satisfied and so, E is measurable.

(ii) The empty set is measurable. (Show empty set is in **A**.)
Since m*(empty set)=0 by definition, then (i) implies that empty set ∈ **A**.

(iii) (Show complements are in **A**.)
Suppose $E \in$ **A**. Then, $m^*(A) = m^*(A \cap E) + m^*(A-E)$, by definition 1.3.2. But $A \cap E = A - E^c$ and $A-E = A \cap E^c$. Hence, $E^c \in$ **A**.

(iv) **Lemma 7:** (Show unions are in **A**.)
Let $E_1 \in$ **A** and $E_2 \in$ **A** be measurable sets. Show $E_1 \cup E_2$ satisfies is measurable.
From defn since E_1 and E_2 are measurable,

$$m^*(A) = m^*(A \cap E_1) + m^*(A - E_1)$$
$$m^*(A-E_1) = m^*((A-E_1) \cap E_2) + m^*((A-E_1) - E_2).$$

Note,

$$(A-E_1) - E_2 = A - (E_1 \cup E_2).$$

Also,

$$[(A-E_1) \cap E_2] \cup [A \cap E_1]$$
$$= [(A-E_1) \cup (A \cap E_1)] \cap [E_2 \cup (A \cap E_1)]$$
$$= A \cap [E_2 \cup E_1].$$

So,

$m^*(A \cap (E_1 \cup E_2)) + m^*(A-(E_1 \cup E_2))$

$= m^*([\, (A-E_1) \cap E_2 \,] \cup [\, A \cap E_1 \,]) + m^*((A-E_1) - E_2)$, and by subadditivity,

$\leq m^*([\, (A-E_1) \cap E_2 \,]\,) + m^*(\, A \cap E_1 \,) + m^*(\, (A-E_1) - E_2 \,)$

$= m^*([\, (A-E_1) \cap E_2 \,]\,) + m^*((A-E_1) - E_2) + m^*(\, A \cap E_1 \,)$, and since $E_2 \in \mathbf{A}$

$= m^*(A-E_1) + m^*(\, A \cap E_1 \,)$, and finally since $E_1 \in \mathbf{A}$

$= m^*(A)$

Hence, we have $m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A-(E_1 \cup E_2))$, which is sufficient to show $E_1 \cup E_2$ is measurable.

(v) (Show differences are in the algebra.)

Let $E_1 \in \mathbf{A}$ and $E_2 \in \mathbf{A}$ be measurable. Show $E_1 - E_2$ is measurable.
Indeed, notice

$$E_1 - E_2 = E_1 \cap E_2^c = (E_1^c \cup E_2)^c.$$

However, complements and unions belong by using (iii) and (iv).

(vi) **Lemma 9** (Show m* is additive when applied to sets in **A**.)

Let $\{E_k\}$ be a sequence of mutually disjoint sets in **A** and denote the union by S_n.

We must show that $m^*(A \cap S_n) = \sum_{k=1}^{m} m^*(A \cap E_k)$. Use induction:

(Basic Step: n=1)

Prove $m^*(A \cap S_1) = \sum_{k=1}^{m} m^*(A \cap E_k) = m^*(A \cap E_1)$, which is true since $S_1=E_1$.

(Induction step)

Assume $m^*(A \cap Sm) = \sum_{k=1}^{m} m^*(A \cap E_k)$ is true for some $m \geq 1$.

Then, show $m^*(A \cap S_{m+1}) = \sum_{k=1}^{m+1} m^*(A \cap E_k)$.

However, by defn,

$m^*(A \cap S_{m+1}) = m^*((A \cap S_{m+1}) \cap S_m) + m^*((A \cap S_{m+1}) - S_m)$.

…and since, $S_{m+1} \cap S_m = S_m$ and $(A \cap S_{m+1}) - S_m = A \cap E_{m+1}$…

$= m^*(A \cap S_m) + m^*(A \cap E_{m+1})$,

and by using the induction hypothesis on the first term,

$= \sum_{k=1}^{m} m^*(A \cap E_k) + m^*(A \cap E_{m+1})$

$= \sum_{k=1}^{m+1} m^*(A \cap E_k)$, as desired.

(vii) (Show m* is completely additive when applied to sets in **A**.)

Let $\{E_n\}$ be an infinite sequence of mutually disjoint sets in **A** and let the union of these be denoted by S. Since m* is monotone and using (vi) on the smaller set $S_n \subseteq S$,

$$m^*(A \cap S) \geq m^*(A \cap S_n) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

Letting $n \to \infty$ noting the left side is independent of n yields

$$m^*(A \cap S) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

The countable subadditivity of m* gives the reverse inequality. So equality must hold.

(viii) **Theorem 10** (Show **A** is a σ-algebra)

Since $A - S \subseteq A - S_n$, then $m^*(A \cap S) \leq m^*(A \cap S_n)$.

Hence,

$$\begin{aligned} m^*(A) &= m^*(A \cap S_n) + m^*(A - S_n) \\ &\geq m^*(A \cap S_n) + m^*(A - S) \\ &= \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A - S). \end{aligned}$$

Letting $n \to \infty$ and using (vii) yields

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A - S) = m^*(A \cap S) + m^*(A - S),$$

which suffices.

(ix) (Show m is a measure)

Easily, noting (viii) gives **A** is a σ-algebra, the restriction m of m* to **A** satisfies all measure properties but complete additivity. To show this, simply use (vii) above with A=S.

Lemma 11: The interval (a,∞) is measurable.

Pf: By the definition of measurable sets, we only need to show for any set A

$$m^*(A) \geq m^*(\, A \cap (a,\infty) \,) + m^*(\, A - (a,\infty) \,).$$

For notational purposes, set

$$B = A \cap (a,\infty) \text{ and } C = A - (a,\infty).$$

If $m^*(A) = \infty$, then the result is trivially true.

Therefore, assume $m^*(A) < \infty$.

Since A has finite measure, for any ε>0 there is a countable collection of open intervals $\{E_n\}$ which cover A and for which

$$\sum_n \lambda(E_n) \leq m^*(A) + \varepsilon.$$

To get back to the original set we want to show measurable, set

$$E_{n,1} = E_n \cap (a,\infty) \text{ and}$$
$$E_{n,2} = E_n \cap (-\infty,a].$$

Each of these are intervals or empty, $B \subseteq \bigcup E_{n,1}$ and $C \subseteq \bigcup E_{n,2}$.
Therefore,

$$m^*(B) \leq m^*(\cup E_{n,1}) \leq \Sigma\, m^*(E_{n,1}), \text{ and}$$

$$m^*(C) \leq m^*(\cup E_{n,2}) \leq \Sigma\, m^*(E_{n,2}).$$

and so

$$m^*(A \cap (a,\infty)) + m^*(A - (a,\infty)) \leq \Sigma\, m^*(E_{n,1}) + \Sigma\, m^*(E_{n,2}) \leq \sum_n \lambda(E_n) \leq m^*(A) + \varepsilon$$

as desired.

HOMEWORK: page 10, #1, 2, 3 (Friedman) and page 64, #10 (Royden)

MEASURABLE FUNCTIONS AND INTEGRATION

Defn: A *measure space* will describe a set X, a σ-algebra of subsets **A** and a measure m, often denoted simply as (X,**A**,m). Notice, for Lebesgue measure, the space $X=R_n$.

If m(X) < ∞, the space is said to be *finite* and σ-*finite* if m is σ-finite.

Note: We will often want to consider the space X=[-∞, ∞], called the extended reals. To do so, topologically we declaring the following sets to be open: (a,b), [-∞,a), (a, ∞] and any union of segments of this type.

Defn: An extended real-valued function f is *measurable* if for any open set M in **R**,

$$f^{-1}(M) = \{x \mid f(x) \in M\}$$

is a measurable set…that is f is measurable provided the inverse image of any open set in the range is a measurable set in the domain X.

Proposition 18 (Theorem 2.1.1 and HW 2.1.4 and 2.1.5): Suppose f is an extended real-valued function defined on a measure space X. Then, the following are equivalent:

1. f is measurable
2. $f^{-1}\{ [-\infty,c) \}$ is measurable for each real number c.
3. $f^{-1}\{ [-\infty,c] \}$ is measurable for each real number c.
4. $f^{-1}\{ (c, \infty] \}$ is measurable for each real number c.
5. $f^{-1}\{ [c, \infty] \}$ is measurable for each real number c.

Further, if these hold, then $f^{-1}\{ c \}$ is measurable for each real number c.

Pf:
(1→2) Assume f is measurable. Notice [-∞,c) is open. Therefore, result holds by definition.
(2→5) Assume $f^{-1}\{ [-\infty,c) \}$ is measurable for each real number c. Since complements of measurable sets are also measurable, then $f^{-1}\{ [c, \infty] \} = f^{-1}\{ [-\infty,c) \}^c$ is measurable.
(5→2) Reverse the roles above to get $f^{-1}\{ [-\infty,c) \} = f^{-1}\{ [c, \infty] \}^c$ is measurable.
(3→4) Again $f^{-1}\{ (c, \infty] \} = f^{-1}\{ [-\infty,c] \}^c$
(4→3) Again $f^{-1}\{ [-\infty,c] \} = f^{-1}\{ (c, \infty] \}^c$
(2→3) Notice $f^{-1}\{ [-\infty,c] \} = \cap f^{-1}\{ [-\infty,c - 1/n) \}$, each of which is measurable.
(3→2) Notice $f^{-1}\{ [-\infty,c) \} = \cup f^{-1}\{ [-\infty,c + 1/n] \}$, each of which is measurable.
(3,5→6) $f^{-1}\{ c \} = f^{-1}\{ [-\infty,c] \} \cap f^{-1}\{ [c, \infty] \}$, each of which is measurable if c is finite
$f^{-1}\{ \infty \} = \cap f^{-1}\{ [n, \infty] \}$, each of which is measurable
$f^{-1}\{ -\infty \} = \cap f^{-1}\{ [-\infty, -n] \}$, each of which is measurable

Defn: A real-valued function f defined on the metric space X is *continuous* if the inverse image of any open set in **R** is open in X.

Theorem 2.1.2: If f is continuous, then f is measurable. (Notice, the σ-algebra on R is created using the open sets.)

HOMEWORK: (Friedman) page 31 #6, 8, 9 (very important), 10

Lemma 2.2.1: If f and g are measurable functions, then the set $E = \{x \mid f(x) < g(x)\}$ is a measurable set.

Pf: The set of all rational numbers is a countable set. Hence, we can write the rationals as an ordered set $\{r_n\}$. Set

$$E_n = \{x \mid f(x) < r_n\} \cap \{x \mid r_n < g(x)\} = f^{-1}(-\infty, r_n) \cap g^{-1}(r_n, \infty).$$

By theorem 2.1.1, the first of these two sets is measurable and the complement of the second is by looking at homework problem 2.1.5.

Hence, E_n is itself measurable for every n and so $E = \cup E_n$ is measurable.

Notation: By writing f(x) + g(x), f(x) g(x) or f(x)/g(x), we will assume no indeterminant forms arise.

Proposition 19 (Theorem 2.2.2): Vertical shifts, sums, differences, multiples and products of measurable functions are measurable functions.

Pf: Suppose f and g are measurable functions and K is any non-zero constant.

<u>Vertical Shifts</u>: f(x) + K

$\{ x : f(x) + K < c \} = \{ x : f(x) < c - K \}$, which is measurable by Proposition 18 (2)

<u>Scalar Multiples</u>: K f(x)

$\{ x : K f(x) < c \} = \{ x : f(x) < c/K \}$, which is measurable by Proposition 18 (2)

<u>Sums</u>: f(x) + g(x)

$\{ x : f(x) + g(x) < c \} = \{ x : f(x) < c - g(x)\}$, which is measurable using Lemma 2.2.1 applied to the measurable functions f(x) and c – g(x).

<u>Differences</u>: f(x) – g(x)

g(x) measurable implies - g(x) is measurable using scalar multiples above

Using sums above, then f(x) – g(x) = f(x) + (-g(x)) is measurable.

<u>Products</u>: f(x) g(x)

Notice, from problem 2.1.9, $| f - g |^2$ and $| f + g |^2$ are measurable.

But $f(x)\, g(x) = \{ | f + g |^2 - | f - g |^2 \}/4$, which is measurable.

General Result: If f and g are measurable and real-valued and H is real and continuous on R^2, then h(x) = H(f(x),g(x)) is measurable.

Pf: Let

$$G_a = \{ (u,v) : H(u,v) < a \},$$

an open subset of R^2. We can write G_a as a union of "open square" intervals E_n, where

$$E_n = \{ (u,v) : a_n < u < b_n,\ c_n < v < d_n \}.$$

Notice, since f and g are measurable, so are

$$\{ x : a_n < f(x) < b_n \} = \{ x : a_n < f(x) \} \cap \{ x : f(x) < b_n \}, \text{ and}$$

$$\{ x : c_n < g(x) < d_n \} = \{ x : c_n < g(x) \} \cap \{ x : g(x) < d_n \}$$

Hence, the composite mapping h(x) satisfies

$$\{ x : h(x) < a \} = \{ x : (f(x),g(x)) \in G_a \} = \cup \{ x : (f(x),g(x)) \in E_n \},$$

which is measurable by above.

Corollary to General Result: The following are measurable:

- f(x) + g(x)
- f(x) g(x)
- f(x)/g(x), provided g(x) is nonzero.

Theorem 2.2.3 (Theorem 20): If $\{f_n\}$ is a sequence of measurable functions, then

- sup $\{f_1, f_2, \dots f_n\}$
- inf $\{f_1, f_2, \dots f_n\}$
- sup $\{ f_n(x) \}$, over the entire infinite sequence
- inf $\{ f_n(x) \}$, over the entire infinite sequence
- lim sup $f_n(x)$
- lim inf $f_n(x)$

are all measurable.

Pf: Notice,

$$\{x \mid \sup f_n(x) \le c \} = \cap \{x \mid f_n(x) \le c \} = \cap f_n^{-1}(-\infty,c],$$

which is the intersection of measurable sets (whether a finite or infinite collections) and therefore measurable. It follows that inf $f_n(x)$ is measurable. By combining sups and infs, we get the rest of the functions are measurable.

Defn: A property P is said to be true *almost everywhere* (a.e.) if the set of points E for which P is not true has measure zero.

Lemma: Any subset of a Lebesgue measurable set of measure zero is Lebesgue measurable.

Pf: Suppose m(E) = 0 and $B \subseteq E$. Then, using monotonicity,

$$m(A \cap B) + m(A - B) \le m(B) + m(A - B) \le m(E) + m(A - B) = m(A - B) \le m(A)$$

and so B is measurable.

Proposition 21: If f is Lebesgue measurable and f = g a.e., then g is Lebesgue measurable.

Pf: Let E be the set where $f \ne g$. By hypothesis, m(E) = 0. Set

$A = f^{-1}((c,\infty))$, which is measurable since f is measurable

$B = [E \cap g^{-1}((c,\infty))]$, which is measurable since it is a subset of E

$C = E \cap g^{-1}((-\infty,c))$, which is measurable since it is a subset of E

So,

$$g^{-1}((c,\infty)) = [A \cup B] - C$$

is measurable being the combination of measurable sets. Therefore, so is the function g.

Corollary 2.2.4: If the sequence $\{f_n\}$ of measurable functions converges to the function g, then g is measurable.

Pf: Apply Proposition 20 noting if lim $f_n(x)$ exist, it is equal to lim sup $f_n(x)$.

Theorem 2.2.5: Let f be a nonnegative measurable functions. Then, there exists a monotone-increasing sequence $\{ f_n \}$ of simple nonnegative functions such that lim $f_n(x) = f(x)$ a.e.

Pf: For n=1,2,3,... divide the y-axis up into "didactic" intervals with endpoints , for $k=0,1,...,n2^n$.
For any given value of x, define:

If $f(x) \geq n$, define $f_n(x) = n$.
If $f(x) < n$ and $(k-1)/2^n \leq f(x) < k/2^n$, for some k, define
$f_n(x) = (k-1)/2^n$ = greatest lower bound endpoint below the actual value of f(x)

Then, $f_n(x)$ is a simple function and $f_{n+1}(x) \geq f_n(x)$.
Case 1: If $f(x) < \infty$ for a given x, then $0 \leq f(x) - f_n(x) \leq 2^{-n}$, which approaches zero as $n\to\infty$.
Case 2: If $f(x) = \infty$ for a given x, then $f_n(x)=n$.
Hence, $f_n(x)$ approaches f(x) as $n\to\infty$.

HOMEWORK: page 35 #3, 6, 7

Defn: A given simple function $f(x) = \sum_{k=1}^{n} s_k \chi_{E_k}(x)$ is said to be *integrable* if $m(E_k) < \infty$, for all k such that a_k is nonzero. The *integral over X* is given by

$$\int s\, dm = \sum_{k=1}^{n} a_k m(E_k)$$

where we use the convention that $0\cdot\infty = 0$. The integral is independent of the (several equivalent) representations of s(x).

If A is any measurable set, then the *Integral of s over A* is given by

$$\int_A s\, dm = \int s \cdot \chi_A \, dm = \sum_{k=1}^{n} a_k m(E_k \cap A)\,.$$

Theorem 2.5.1: Let f and g be integrable simple functions and a and b be real numbers. Then,

1. $\int \{a f + b g\}\, dm = a \int f\, dm + b \int g\, dm$
2. If $f \geq 0$ a.e., then $\int_E f\, dm \geq 0$.
3. If $f \geq g$ a.e., then $\int_E f\, dm \geq \int_E g\, dm$.
4. $| f |$ is integrable and $| \int f\, dm| \leq \int | f |\, dm$
5. $\int |f + g|\, dm \leq \int |f|\, dm + \int |g|\, dm$
6. $\alpha \leq f \leq \beta$ a.e. on a measurable set E with $m(E) < \infty$ yields $\alpha\cdot m(E) \leq \int_E f\, dm \leq \beta\cdot m(E)$.
7. If $f \geq 0$ a.e. and E and F are measurable sets such that $E \subseteq F$, then $\int_E f\, dm \leq \int_F f\, dm$.
8. If E is a disjoint union of measurable sets E_k, then $\int_E f\, dm = \Sigma \int_{E_k} f\, dm$

Remark: The idea of a sequence of functions being "close" to another function can be defined in several ways. Usually, given an x-value, one thinks of close as the sequence of y-values $f_n(x)$ getting closer to its limit f(x).

Defn: A measurable function f is said to be *a.e. real-valued* is the set $\{x \mid |f(x)| = \infty\}$ has measure zero.

Defn: A sequence $\{f_n\}$ is *convergent in measure* if there is a measurable function f such that for any $\in>0$,

$$\lim m[\{ x : |f_n(x) - f(x)| > \in \}] = 0.$$

Theorem: If $\{f_n\}$ converges in measure to both f and g, then f=g a.e. and both are real-valued a.e.

Defn: A sequence of measurable functions f_n is said to be a *Cauchy sequence in the mean* if

$$\int | f_n - f_m| \, dm \to 0 \text{ as } n,m \to \infty .$$

Lemma 2.5.2: If f_n is a sequence of integrable simple functions that is Cauchy in the mean, then there is an a.e. real-valued, measurable function f such that f_n converges in measure to f.

Pf: Let $\in>0$. Choose $E_{n,m}= \{ x \mid |f_n(x) - f_m(x)| > \in |$.
Since $f_n(x) - f_m(x)$ is integrable, $E_{n,m}$ has finite measure.
Theorem 2.5.1 implies $\int | f_n - f_m| \, dm \geq \in m(E_{n,m}) \geq 0$
Since the sequence is Cauchy, the integral approaches zero and hence $m(E_{n,m}) \to 0$ as $n,m\to\infty$.
Hence f_n is Cauchy in measure and so by Corollary 2.4.4, f_n is convergent in measure.

HOMEWORK: page 42, #1, 2, 3

Consider the following conditions:

- C1: $\{f_n\}$ is a Cauchy sequence in the mean
- C2: $\lim f_n = f$, a.e.
- C3: $\{f_n\}$ converges in measure to f

Defn 2.6.1: f is said to be *integrable* if there exists a sequence $\{f_n\}$ of integrable simple functions such that C1 and C2 hold.

(Royden's definition) A nonnegative measurable function f is said to be integrable over the measurable set E provided $\int_E f \, dm < \infty$. A general measurable function f is said to be integrable provided $f = f^+ - f^-$ and $\int_E f^+ \, dm < \infty$ and $\int_E f^- \, dm < \infty$

Theorem 2.6.1: f is integrable if and only if C1 and C3 hold.

Pf: Suppose C1 and C2 hold. Lemma 2.5.2 implies $\{f_n\}$ converges in measure to an a.e. real-valued, measurable function g.
Theorem 2.4.3 and 2.3.1 imply that there is a subsequence $\{f_{n,k}\}$ of $\{f_n\}$ that converges a.e. to g.
C2 implies g=f, a.e. Hence, $\{f_n\}$ converges in measure to f.
Conversely, suppose C1 and C3 hold. Then, there is a sequence $\{g_n\}$ of integrable simple functions such that $\{g_n\}$ is Cauchy in the mean and converges in measure to g.
Theorems 2.4.3 and 2.3.1 imply there is a subsequence $\{g_{n,k}\}$ of $\{g_n\}$ that is convergent to f a.e.
Denote this subsequence by $\{f_k\}$. Then, $\{f_k\}$ satisfies C1 and C2.

Result: If f is integrable, then f is a.e. real-valued. (See result preceding Theorem 2.4.1.)

Defn 2.6.2: Let f be an integrable function and let C1 and C2 hold. The *integral* of f is defined to be the number $\lim \int f_n \, dm$ and is denoted by $\int f \, dm$. Hence, $\int f \, dm = \lim \int f_n \, dm$.

Theorem 2.6.2: The definition of $\int f \, dm$ is independent of the sequence $\{f_n\}$ chosen.
Proof: See lemmas in text, pages 43-46.

Defns (2.6.2, 2.6.5, 2.6.6): Suppose E is a measurable set and f an integrable function. Then, the integral of f over E is defined by

$$\int_E f \, dm = \lim \int \chi_E f_n \, dm$$

If we are using a Lebesque measure space, we often denote the Lebesque integral of f as

$$\int_E f(x) \, dx$$

If f is a non-negative measurable function and not integrable on the set E, then we say that

$$\int_E f \, dm = \infty.$$

HOMEWORK: page 47 #1, 2, 3, 7

Theorem 2.7.1: Let f and g be integrable functions and a and b be real numbers. Then,
(i) $\int (a f + b g) \, dm = a \int f \, dm + b \int g \, dm$
Pf: Take limits in 2.5.1.

(ii) If $f \geq 0$ a.e., then $\int f \, dm \geq 0$.

(iii) If $f \geq g$ a.e., then $\int f \, dm \geq \int g \, dm$.

(iv) $| f |$ is integrable and $| \int f \, dm | \leq \int | f | \, dm$
Pf: Note $f \leq | f |$ and $-f \leq | f |$. Apply (iii).

(v) $\int | f + g | \, dm \leq \int | f | \, dm + \int | g | \, dm$

(vi) $m \leq f \leq M$ a.e. on a measurable set E with $m(E) < \infty$ yields $m \, m(E) \leq \int_E f \, dm \leq M \, m(E)$.

(vii) If $f \geq 0$ a.e. and E and F are measurable sets such that $E \subseteq F$, then $\int_E f \, dm \leq \int_F f \, dm$.

(viii) If $f \geq m > 0$ on a measurable set E, then $m(E) < \infty$.
Pf: Assume m(E) is infinite.
By problem 2.6.2, E is σ-finite.
Hence, there exist a monotone-increasing sequence of sets E_k with $m(E_k) < \infty$ and $\lim E_k = E$.
By (vii), $\int_E f \, dm \geq \int_{E_k} f \, dm \geq m \, m(E_k)$ which approaches *oo*. Contradiction.

Defn 2.7.1: A sequence $\{f_n\}$ of integrable functions is said to be a *Cauchy sequence in the mean* if

$$\int | f_n - f_m | \, dm \to 0$$

as n and m get large. If there is an integrable function f such that

$$\int | f_n - f | \, dm \to 0$$

as n gets large, then we say that $\{f_n\}$ *converges in the mean* to f.

Result: If $\{f_n\}$ is convergent in the mean to f, then it is also Cauchy in the mean.

Result: Suppose f and g are measurable on the measurable set E and $0 \leq g(x) \leq f(x)$ over E. Then, f integrable over E implies g is also integrable over E.

Pf: $\infty > \int_E f\, dm = \int_E (f - g)\, dm + \int_E g\, dm$. Since $f - g \geq 0$, then $\int_E (f - g)\, dm > 0$ and so both terms on the right must also be finite.

Result: Assume that f is nonnegative and integrable over a measurable set E. Then, for any $\epsilon>0$, there is a corresponding $\delta>0$ such that for every set $A \subseteq E$ with $m(A) < \delta$ we get $\int_A f\, dm < \epsilon$.

Result: Let $\{f_n\}$ be a sequence of simple functions convergent to f and Cauchy in the mean. Then, $\lim \int f_n\, dm$ exists.

Pf: Consider $| \int f_n\, dm - \int f_m\, dm | \leq \int |f_n - f_m|\, dm$ which approaches zero since convergent implies Cauchy.

Defn: Convergence in measure – see Friedman. f_n converges to f "in measure" provided the collection of x-values for which the sequence $f_n(x)$ does NOT converge to f(x) has measure zero.

Theorem 2.7.2: If $\{f_n\}$ is a sequence of integrable functions that converges in the mean to an integrable function f, then $\{f_n\}$ converges in measure to f.

Pf: Let $\epsilon>0$ be given and define $E_n = \{ x \mid |f_n(x) - f(x)| \geq \epsilon\}$.

By Theorem 2.7.1, $m(E_n)<\infty$ and

$$\int_E | f_n - f |\, dm \geq \int_{E_n} | f_n - f |\, dm \geq \epsilon\, m(E_n).$$

Hence, $m(E_n)$ approaches zero as n gets large.

Theorem 2.7.3: If f is an a.e. nonnegative, integrable function, then $\int_E f\, dm = 0$ if and only if f=0 a.e. on E.

Pf: If f=0 a.e., then HW 2.6.1 with g=0 everywhere yields $\int_E f\, dm = 0$.

On the other hand, if $\int_E f\, dm = 0$, then there exists a sequence $\{f_n\}$ that is Cauchy in the mean and this is convergent in measure to f.

Since $f \geq 0$, then the same is true for $|f_n|$. So,

$$\lim \int_E f_n\, dm = \int_E f\, dm = 0.$$

Hence, $\{f_n\}$ converges in the mean to zero.

Theorem 2.7.2 implies that $\{f_n\}$ converges in measure to zero and thus f=0 a.e.

Theorem 2.7.4: Let f be measurable and E a set of measure zero.
Then, f is integrable on E and $\int_E f\, dm = 0$.

Pf: Notice that $\chi_E f = 0$ a.e. By Problem 2.6.1, $\chi_E f$ is integrable and $\int \chi_E f\, dm = 0$.

Theorem 2.7.5: Let f be an integrable function that is positive everywhere on a measurable set E.
If ,/`$_E$ f dm = 0, then m(E)=0.

Pf: Let $E_n = \{x \in E \mid f(x) > 1/n \}$.

Then, $\{E_n\}$ is a monotone increasing sequence of sets and $E - \bigcup E_n$ has measure zero.

Hence, $m(E) = \lim m(E_n)$.

Since $m(E_n)$ is finite, $0 = \int_E f\, dm \geq \int_{E_n} f\, dm \geq m(E_n)/n \geq 0$.

Thus, $m(E_n)=0$ for all n and so m(E)=0.

Theorem 2.7.6: Let f be an integrable function. If $\int_E f\,dm = 0$ for every measurable set E, then f=0 a.e.

Pf. By Theorem 2.7.5, the set where f(x)>0 has measure zero.
Similarly, the set where f(x)<0 has measure zero.
Hence, f=0 a.e.

HOMEWORK: page 50 #2, 3, 4, 6

Littlewood's Principle (Proposition 23): Let E be measurable with $m(E)<\infty$ and $\{f_n\}$ measurable on E such that for each x in E, $f_n(x)\to f(x)$, for some real-valued function f. Then, for any $\epsilon > 0$ and $\delta > 0$, there is a measurable set $A\subseteq E$ with $m(A) < \delta$ and an integer N such that for all $x\notin A$ and $n \geq N$,

$$| f_n(x) - f(x) | < \epsilon.$$

Pf: See page 73 of Royden. The conclusion implies that the set of x-values A for which $f_n(x)$ and f(x) may NOT be close is arbitrarily small

Bounded Convergence Theorem: Let $\{f_n\}$ be a sequence of measurable functions over a set E of finite measure that converges on E to a measurable function f. Further, suppose there exists an constant M such that $|f_n(x)| \leq M$, for all n. Then,

$$\lim_{n\to\infty} \int_E f_n\,dm = \int_E f\,dm.$$

Pf: Using Littlewood's Principle with $m(A) < \epsilon/(4M)$ yields $| f_n(x) - f(x) | < \epsilon/(2m(E))$.
Therefore,

$$\begin{aligned} \left| \int_E f_n\,dm - \int_E f\,dm \right| &= \left| \int_E (f_n - f)\,dm \right| \\ &\leq \int_E | f_n - f |\,dm \\ &= \int_{E-A} | f_n - f |\,dm + \int_A | f_n - f |\,dm \\ &\leq \int_{E-A} \epsilon/(2m(E))\,dm + \int_A 2M\,dm \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

The conclusion follows

Fatou's Lemma (Theorem 9): Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions which converge to f a.e. on E. Then, $\int_E f\,dm \leq \liminf \int_E f_n\,dm$.

Pf: WOLOG, we can assume the convergence is everywhere since we can throw out integrals over sets of measure zero.
Choose h to be an arbitrary but bounded and measurable function with $h \leq f$ and h(x) = 0 outside a set E' with $m(E') < \infty$. For each n, define $h_n(x) = \min\{ h(x), f_n(x) \}$.
Then, h_n is bounded and equals zero outside E'.
Since $h \leq f$, $h_n(x)\to h(x)$ for each x in E'.
The Bounded Convergence Theorem implies

$$\int_E h\,dm = \int_{E-E'} h\,dm + \int_{E'} h = \int_{E'} h = \lim \int_{E'} h_n\,dm = \liminf \int_{E'} h_n\,dm \leq \liminf \int_{E'} f_n\,dm.$$

Taking the supremum on both sides over all $h \leq f$ and applying the definition of integral implies the result.

Monotone Convergence Theorem (MCT): Suppose $\{f_n\}$ is an increasing sequence of nonnegative measurable functions which converge to f a.e. on E. Then $\int_E f\,dm = \lim \int_E f_n\,dm$.
Pf: By Fatou's Lemma, we have $\int_E f\,dm \leq \liminf \int_E f_n\,dm$.
However, since the sequence is increasing, then $f_n(x) \leq f(x)$ and so $\int_E f_n\,dm \leq \int_E f\,dm$.
Take the lim sup of both sides to get

$$. \limsup \int_E f_n\,dm \leq \int_E f\,dm \leq \liminf \int_E f_n\,dm$$

and so the limit exists and the result follows.

Corollary 11: If u_n is a sequence of nonnegative measurable functions and $f = \sum u_n$, then $\int f = \sum \int u_n$.

Pf: Take f_n to be the sequence of partial sums.

Corollary 12: Let f be a nonnegative function and $\{E_k\}$ disjoint and measurable with $E = \cup E_k$. Then

$$\int_E f = \sum \int_{E_k} f.$$

Pf: Take $u_k(x)= f(x)\, \chi_{E_k}(x)$.

Reminder: For general measurable functions $f = f^+ - f^-$. To integrate f, apply the integral to these component parts.

Lebesgue Dominated Convergence Theorem (LDCT) – page 91

Generalized Dominated Convergence Theorem (G-LDCT) – page 92

METRIC AND L^p SPACES

Defn: A *metric space* will describe a set X with a function ρ so that for any two points (x,y), there corresponds a real number $\rho(x,y)$ such that:

- $\rho(x,y) \geq 0$ and $\rho(x,y)=0$ if and only if x=y
- $\rho(x,y) = \rho(y,x)$
- $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$, for any z

Examples of metric spaces:

1. n-dimensional reals $\mathbb{R}^n$ using the Euclidean metric
2. sequence space ℓ^∞
3. sequence space ℓ^1
4. sequence space c
5. C([a,b]) = continuous functions defined on the interval [a,b] with $\rho(f,g) = \max | f(x)-g(x) |$

L^p Spaces: Given a positive number p, denote by $L^p(X)$ the collection of functions defined on X such that $|f|^p$ is integrable. Define the "p-norm" of a given function f to be:

$$|| f ||_p = \{ \int |f|^p \, dm \}^{1/p}$$

Similarly, define $L^\infty(X)$ to be the collection of measurable and essentially bounded functions.

Define the "∞-norm" of a given function to be:

$$|| f ||_\infty = \text{essential supremum } |f|$$

Holder's Inequality: Assume 1/p + 1/q = 1. If $f \in L^p(X)$ and $g \in L^q(X)$, then $f\,g \in L^1(X)$ and

$$|| f ||_1 \leq || f ||_p \, || g ||_q$$

Pf: See page 96

Minkowski's Inequality: Let $1 \leq p \leq \infty$. Then, $f,g \in L^p(X)$ implies $f+g \in L^p(X)$ and

$$|| f+g ||_p \leq || f ||_p + || g ||_p.$$

Pf: See page 97

Theorem 3.2.3: If $1 \leq p \leq \infty$, then $L^p(X)$ is a complete metric space.

Pf: See page 98

Extra Material
Friedman Section 1.4-1.7

Defn: The class of sets **K** is a *sequential convering class* if empty set∈**K** and if for any set A∈**K**, there is a sequence of sets E_n∈**K** such that A $\underline{C}$ U E_n.

Construction of Outer Measures: The previous work shows that if one can construct an outer measure, then restricting the domain of the outer measure to measurable sets yields a measure for that collection of sets. However, this assumes that one is able to start with some given outer measure. Such an outer measure can be constructed from any collection of sets by using the following:
Suppose we have any nonnegative set function □with domain **K** such that □(empty set)=0. For each set A of X, create the set function

$$m^*(A) = \inf\{ \Box\ \Box(En) \mid E_n \in \mathbf{K},\ A\ \underline{C}\ U\ E_n \}$$

Theorem 1.4.1: (Show that the method of constructing an outer measure above indeed yields an outer measure.)
Pf:
(i) Obviously the domain consists of all subsets of X by the way m* is defined.
(ii) m* is nonnegative since l is nonnegative
(iv) m* is monotone since if A$\underline{C}$B, then any covering of B also covers A. Hence, there could be a smaller infinum for A and thus a smaller value for m*(A).
(iii) We must show m* is countably additive:

Let A_n be any sequence of sets and take any ∈ > 0.
Since **K** is a convering class, for each A_n, there is a sequence of covering sets E_{nk} such that

$$m^*(A) + \in/2n \geq \Box\ \Box(E_{nk}).$$

So, U A_n $\underline{C}$ U E_{nk} and thus by monotonicity, m*(U A_n) ≤ m*(U E_{nk}).
Hence, m*(U A_n) ≤ m*(U E_{nk}) ≤ m*(A) + ∈/2n = m*(A) + ∈. Since ∈ is arbitrary, we have m*(U A_n) ≤ m*(A) as desired.

(v) Since m* is nonnegative, the the smallest it can be is zero. Since **K** is a convering class, empty set∈**K** and so A=empty set can be convered most simply by itself with □(empty set)=0.

HOMEWORK: page 12, #2, 3

Defn: A measure m̩ with domain **A** is said to be *complete* if N $\underline{C}$ E∈**A** and m(E)=0 implies N∈**A**.

Result: The measure constructed in Theorem 1.3.1 is complete.

Pf: By (i) of proof, any set with outer measure zero is measurable and thus in **A**.

Theorem 1.5.1: Any measure can be extended to be a complete measure.

Lebesgue Measure: Let X = $\mathbf{R}_n$ = { (x_1, x_2 x_3 ... x_n) where each component is a real number }.The collection **K** of open intervals forms a sequential convering class of X. Define the set function □by □(empty set)=0 and for each non-null interval I (see defn on page 13)

□(I) = □ (b_k - a_k) = product of the lengths of each component's interval width.

From Theorem 1.4.1, this set function yields an outer measure which we call Lebesgue outer measure. By Theorem 1.3.1, the restriction of this outer measure yields a measure (which is complete) called Lebesgue measure. The measurable sets are called Lebesgue measurable sets.

HOMEWORK: page 14, #1, 2, 3

Defn: A *metric* □ is a function defined on a set of points X satisfying positive definiteness, symmetry and the triangle inequality. The set of points X together with the metric □ is called a *metric space*. Given a metric □, the *distance* between two sets A and B is defined by □(A,B) = inf { □(x,y) } $x \in A$ and $y \in B$ }. If either set is a single point (say A={x}), we write this distance as □(x,B). For a given set A, its *diameter* is given by □(A) = sup { □(x,y) } $x \in A$ and $y \in A$ } and the set is *bounded* if □(A) is finite.

Defns: For any element x and ∈>0, an *open ball* is the set B(x, ∈) = { y | □(x,y) < ∈ } with x called the *center* and ∈ called the *radius*. A *closed ball* allows equality in above and is denoted `B(x, ∈). A sequence x_n is said to be *convergent* to y if $r(x_n,y)$-->0 as n-->*oo*. Open sets, closure, closed set, interior, Cauchy sequences, complete metric space....see page 17.

HOMEWORK: page 17, #2, 3, 4, 6

Extra Material
Friedman Sections 2.3-2.6

Defn: A sequence $\{f_n\}$ of a.e. real-valued, measurable functions is said to *converge almost uniformly* to a measurable function f if for any ∈>0, there exists a measurable set E such that m(E) < ∈ and $\{f_n\}$ converges to f uniformly on X - E.

Theorem 2.3.1: If a sequence $\{f_n\}$ of a.e. real-valued, measurable functions converges almost uniformly to a measurable function f, then $\{f_n\}$ converges to f a.e.

Pf: Since $\{f_n\}$ converges almost uniformly to f, for any integer m>0 there is a set E_m such that $m(E_m) < 1/m$ and $\{f_n\}$ converges to f uniformly on X - E_m. Hence, $\{f_n\}$ converges to f on F = U ($X-E_m$) = X - ∩ E_m.
But m(X - F) = m(∩ E_m) $\leq m(E_m) < 1/m$ for any positive integer m. Thus m(X - F) = 0 and so $\{f_n\}$ converges to f a.e.

Theorem 2.3.2: (Egoroff's Theorem) Let X be a finite measure space. If a sequence $\{f_n\}$ of a.e. real-valued, measurable functions converges a.e. to f, then $\{f_n\}$ converges to f almost uniformly.

Pf: Since f and $\{f_n\}$ are real-valued a.e., then it is sufficient to assume that all functions are real-valued everywhere. For positive integers k and n, define

$$E_{n,k} = \cap_{m=n...}\{x \mid |f_m(x) - f(x)| < 1/k \}.$$

Notice, as n increases, the sets $E_{n,k}$ form a monotone increasing sequence of sets. Since $\{f_n\}$ converges a.e. to f, then lim $E_{n,k}$ = E, where E is a set such that X-E has measure zero. By Theorem 1.2.1,

$$\lim m(x- E_{n,k}) = m(X-E) = 0.$$

Hence, for any ∈>0 there is an integer n_k such that $m(X - E_{n,k}) <$ ∈$/2^k$, if $n \geq n_k$.
If F = ∩ $E_{n,k}$, then F is measurable and

$$m(X - F) = m(X - \cap E_{n,k}) = m(U (X-E_{n,k})) \leq □\ m(X-E_{n,k}k) < ∈.$$

Hence, on the set F, the sequence $\{f_n\}$ is uniformly convergent to f.

Theorem 2.4.1: If a sequence $\{f_n\}$ of a.e. real-valued measurable functions converges almost uniformly to a measurable function f, then $\{f_n\}$ converges in measure to f.

Pf: For and ∈>0 and □>0, there is a set E with m(E)<□ such that $|f_n(x) - f(x)| <$ ∈, for all x ∈X-E and n sufficiently large. This implies convergence in measure.

Corollary 2.4.2: If $m(X) < \infty$, then any sequence $\{f_n\}$ of a.e. real-valued, measurable functions that converges a.e. to an a.e. real-valued, measurable function f is also convergent to f in measure.

Pf: Apply Theorem 2.4.1 and Egoroff's Theorem.

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Lemma 2.8.1: If $\{f_n\}$ is a sequence of integrable *simple* functions yielding the integrable function f, then $\lim_{n \to \infty} \int | f_n - f | \, dm = 0$

Pf: Consider the sequence $\{g_k\} = \{ | f_n - f_k | \}$, for any positive integer n.

Notice that $| |a| - |b| | \leq | a - b |$.

Hence,

$$\int | g_m - g_k | \, dm = \int | | f_n - f_m | - | f_n - f_k | | \, dm \leq \int | f_m - f_k | \, dm \to 0$$

as m and k get large, since $\{f_n\}$ is Cauchy in the mean.

Thus, $\{g_k\}$ is also Cauchy in the mean. Further, g_k converges to $| f_n - f |$ a.e.

Applying the definition of integral yields the result.

Theorem 2.8.2: If $\{f_n\}$ is a sequence of integrable functions which are Cauchy in the mean such that $\lim f_n = f$, an integrable function, then $\lim_{n\to\infty} \int f_n \, dm = \int f \, dm$.

Pf: By Lemma 2.8.1, for each n, there is a sequence of integrable simple functions $\{f_{n,k}\}$ such that $\lim_{k\to\infty} \int | f_n - f_{n,k} | \, dm = 0$.

Hence, for each n, there is a term ff_n of the sequence $f_{n,k}$ such that $\int | f_n - ff_n | \, dm < 1/n^2$.

The proof of Theorem 2.7.2 with $\in = 1/n$ yields $m\{ x \mid | f_n(x) - ff_n(x) | \geq 1/n \} < 1/n$. Hence, $\{ff_n \}$ is Cauchy in the mean and converges in measure to f. Therefore, f is integrable and

$$\int f \, dm = \lim_{n\to\infty} \int ff_n \, dm = \lim_{n\to\infty} \int f_n \, dm$$

Theorem 2.8.3: If $\{f_n\}$ is a sequence of integrable functions that is Cauchy in the mean, then there is an integrable function f such that $\{f_n\}$ converges in the mean to f.

Pf: Extending the proof of Lemma 2.5.2 applied to any integrable function, we conclude that $\{f_n\}$ is convergent in measure to a measurable function f. Theorem 2.8.2 implies that f is integrable. Hence, the sequence $\{| f - f_n |\}$ is a sequence of integrable functions that is Cauchy in the mean and that converges in measure to 0.

Theorem 2.8.1 implies $\lim_{n\to\infty} \int | f - f_n | \, dm = 0$

Defn 2.8.1: A real-valued set function λ is said to be *absolutely continuous* is for any $\in>0$, there exists a number $\Delta>0$ such that for any measurable set E with $m(E)< \Delta$, $|\lambda(E)|< \in$.

Theorem 2.8.4: Let f be an integrable function and let I be the set function defined by □(E) = for all the measurable sets E. Then, □ is completely additive and absolutely continuous and is called the *indefinite integral* of f.

Pf: see text.

Theorem 2.9.1: (*Lebesque's Bounded Convergence Theorem* - **LBCT**) Let $\{f_n\}$ be a sequence of integrable functions that converges either in measure or a.e. to a measurable function f. Suppose there exists an integrable function g such that $|f_n(x)| \leq g(x)$ a.e. for all n.

Then, f is integrable and $\lim_{n \to \infty} \int | f - f_n | \, dm = 0$.

Pf:

<u>Case I</u>: Suppose $\{f_n\}$ converges to f in measure.

We must show that $\{f_n\}$ is a Cauchy sequence in the mean. If so, then by Theorem 2.8.3 an integrable function h exists such that $\lim_{n\to\infty} \int | h - f_n | dm = 0$. Theorem 2.7.2 implies then that $\{f_n\}$ converges in measure to h. Since f is also the limit in measure of $\{f_n\}$, we have f = h a.e. and so f is integrable. Replacing h with f gives the result.

To show $\{f_n\}$ is a Cauchy sequence in the mean, see Friedman, page 55.

<u>Case II</u>: Suppose $\{f_n\}$ converges a.e. to f. Show $\{f_n\}$ also converges in measure to f and apply Case I.
Set $N = \{x \mid |f(x)|>g(x) \text{ or } |f_n(x)|>g(x) \}$ and for any $\in>0$, set $E_n = U_{k=n..} \{ x \mid |f_k(x) - f(x) | \geq \in\}$.
Notice, $\{ x \mid |f_n(x) - f(x) | \geq \in\} \subseteq E_n$.
Then, $E_n \subseteq \{x \mid g(x) > \in/2 \} \cup N$.
Since g is integrable, HW 2.7.2 implies that $m(E_n)$ is finite. Since f_n converges to f a.e., $m(E_n)=0$.
Therefore, by Theorem 1.2.1, $\lim m(E_n)=0$ and so $\{f_n\}$ converges to f in measure.

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Theorem 2.10.1: Let f and g be measurable. If $|f| \leq g$ a.e. and g is integrable, then f is integrable.
Pf: HW problem 2.6.3 implies that f is integrable if and only if |f| is integrable. Indeed,

=> Assume |f| is integrable. Then, $| \int f\, dm | \leq \int |f|\, dm$, which is finite by assumption.

<= Assume f is integrable. Then, $\int |f|\, dm = \int_A |f|\, dm + \int_B |f|\, dm \leq \int_A f^+\, dm + \int_B f^-\, dm$ both of which are finite, where $A=\{x: f(x)\geq 0\}$ and $B=\{x:f(x)<0\}$.

So, if we can show |f| is integrable, then so is f.
However, by T2.2.5, we can approximate any measurable function f with an increasing sequence $\{h_n\}$ of simple functions. So, $h_n \leq |f| \leq g$ implies $\{h_n\} \leq g$. Since g is integrable, using HW 2.7.2 implies the $\{h_n\}$ are also integrable. Apply the LBCT to complete.

Theorem 2.10.4: *(Lebesque Monotone Convergence Theorem = LMCT)* Let $\{f_n\}$ be a monotone increasing sequence of non-negative integrable functions and let $f(x) = \lim f_n(x)$.
Then, $\lim_{n\to\infty} \int f_n\, dm = \int f\, dm$.
Pf: If f is integrable, then $f_n \leq f$ easily implies $\int f_n\, dm \leq \int f\, dm$, for all n and so

$$\lim_{n\to\infty} \int f_n\, dm \leq \int f\, dm.$$

If f is not integrable, then is infinite and so the inequality still holds. It remains to show that *equality* holds.
If $\lim_{n\to\infty} \int f_n\, dm$ is infinite, then equality will hold. So consider the case when this limit is finite.
Show C1 and C2 hold. By hypothesis C2 holds. Hence, we must show the the sequence f_n is Cauchy in the mean. Consequently, consider f_n and f_m, where we'll assume that $m\geq n$.
By monotonicity, $f_m \geq f_n$, and so $f_m - f_n \geq 0$.
Hence, $\int | f_m - f_n |\, dm = \int (f_m - f_n)\, dm = \int f_m\, dm - \int f_n\, dm$ which approaches zero as m and n get large.
Thus, C1 and C2 hold. By Theorem 2.8.2, f is integrable and the result holds.

Theorem 2.10.5: *(Fatou's Lemma)* Let $\{f_n\}$ be a sequence of nonnegative integrable functions and let $f(x) = \liminf f_n(x)$. Then

$$\liminf_{n\to\infty} \int f_n\, dm \leq \int f\, dm.$$

Thus, if $\liminf_{n\to\infty} \int f_n\, dm$ is finite, f is integrable.
Pf: If $\liminf_{n\to\infty} \int f_n\, dm$ is infinite, the inequality is obviously true. So, suppose it is finite.

Set $g_n = \inf_{j \geq n} f_j(x)$. Then, $\{g_n\}$ is a monotone-increasing sequence of non-negative integrable functions and $g_n \leq f_n$. Hence, $\lim \int g_n \, d\mu \leq \liminf \int f_n \, d\mu$ which is finite.
Since $\lim g_n = \lim f_n = f$, apply the LMCT to see that f is integrable and $\lim_{n \to \infty} \int g_n \, d\mu = \int f \, dm$. Combine with the above inequality.

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