Lecture Notes
Graph Theory
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Based upon "Graph Theory", Bondy and Murty, Springer, 2008

This text is an update and extension of a classic but out-of-print text which can be downloaded at http://www.ecp6.jussieu.fr/pageperso/bondy/books/gtwa/gtwa.html .

Chapter Two

Subgraphs

Natural subgraphs
Deleting an edge, G \ e = the graph obtained by removing only that edge and leaving the vertex set unchanged.
Deleting a vertex, G – v = the graph obtained by removing a vertex and any incident edges.

Copy of F in G, embedding of a graph F in G

Super-graph

Theorem 2.1. Let G be a graph in which all vertices have degree at least two. Then G contains a cycle.
Pf: (An example of proof by contradiction of a supposed maximum.)
If G contains a loop, then this is also a cycle and the result is true.
So, assume that G is loopless and let P be a path contained in G which is as long as possible.
Denote this path by its vertices $v_1v_2\ldots v_{k-1}v_k$.
By assumption, $d(v_k) > 1$ and so there is another edge e leaving v_k other than the edge along the path P.
If e is incident with another vertex not in P, then we can extend P by this one more edge which contradicts the assumption that P is as long as possible.
So e must be incident with one of the vertices in P, say v_j.
Then, a loop can be obtained by following P from v_j to v_k and then back along e.
(Note, v_{k-1} could be this other vertex since there could be parallel edges.)

Maximal subgraph (maximal path)
Minimal subgraph
Maximum cycle, circumference, girth

Acyclic graph

Corollary. An acyclic graph must have at least one vertex of degree smaller than 2.

Partially ordered sets (posets)

Pigeonhole Principle – page 43

Homework – page 43
#1, 2, 3, 4, 7, 11, 15, 17, 21

Spanning Subgraph, G \ S, where S is a set of edges
Spanning Supergraph, G + S
Join of graphs, $G \vee H$, obtained by adding edges between all vertices of G with those of H
Hamilton paths and cycles – traverse every vertex without repeats

Underlying Simple Graph, obtained by removing all loops and duplicate edges
Symmetric Difference, $G \triangle H$, assuming G and H are on the same vertex set, by removing all edges common to both G and H.

Theorem 2.3 (Redei's Theorem) Every tournament has a directed Hamilton path. – page 48
Pf: Certainly true for a one vertex tournament and also easily true for a two vertex tournament. By induction…

Theorem 2.4. Every loopless graph G contains a spanning bipartite subgraph F such that for all vertices v, $d_F(v) \geq 0.5\, d_G(v)$.
Pf. By contradiction…

Induced subgraph – obtained by deleting vertices, G – S

Theorem 2.5. Every graph with average degree at least 2k (k a positive integer) has an induced subgraph with minimum degree at least k+1.

Weighted graphs, w(e)

Homework – page 51
#1, 2, 4, 6, 7, 11, 12, 19

Edge cut
$\delta(X)$ = edges associated with an edge cut consisting of all edges with one endpoint in X
G(V,E) is bipartite if $\delta(X) = E$
G(V,E) is connected if $\delta(X)$ is nonempty for all proper vertex subsets X
If G is loopless, then $|\delta(v)| = d(v)$

Homework – page 63
#1, 4, 5