Advanced Calculus Notes
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Based upon "An Introduction to Analysis", Wade, 3rd edition

CHAPTER 4

4.1 – Derivative: A given function f is *differentiable* at the point a provided

$$(1)\ \ f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

exists. If so, then f′(a) is called the *derivative* of f(x) at x=a. Notice that h→0 is equivalent to x→a and so the definition formula can be rewritten (using h = x - a) as

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

Further if we set

$$(2)\ \ F(x) = \frac{f(x)-f(a)}{x-a}$$

then $f'(a) = \lim_{x\to a} F(x)$. Hence, if F(x) is continuous at x=a, then f′(a) = F(a).

4.2 – Theorem: f differentiable ⇔ ∃ continuous F such that f(x) = F(x)(x-a) + f(a) and f′(a) = F(a)

Pf: If f is differentiable, then f′(a) exists. So, define F(x) using (2) if x≠a and F(a) = f′(a). The result follows.

Conversely, if F exists, then take the limit as x approaches a to get the alternate form of the deriviative.

4.3 – Alternate Characterization of Differentiability: f differentiable ⇔ ∃ T(x) = mx such that

$$(4)\ \ \lim_{h\to 0}\left|\frac{f(a+h)-f(a)-T(h)}{h}\right| = 0$$

Pf: If f is differentiable, set T(x) = f′(a) x.

If (4) holds for some T(x) = mx, then - aiming toward (1) - we have

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} - m = \lim_{h\to 0}\frac{f(a+h)-f(a)-mh}{h} = \lim_{h\to 0}\frac{f(a+h)-f(a)-T(h)}{h} = 0$$

Hence, the limit exists and we get m = f′(a).

4.4 – Differentiability implies Continuity: f differentiable ⇒ f continuous

Pf: Assume f is differentiable. Then, using (2) yields

$$\lim_{x\to a} f(x) = \lim_{x\to a} F(x)(x-a) + f(a).$$

Since F(x) is continuous, then applying the limit theorems yields the result.

4.6 – Differentiability on Intervals and Continuously Differentiable

Homework: page 90 #4, 6, 8
In Class: 1, 3, 5

4.10 – Rules for Derivatives

4.11 – Chain Rule: f and g differentiable implies $g\circ f$ is differentiable

Pf: By Theorem 4.2, consider F(x) and G(x) given by

$$f(x) = F(x)(x-a) + f(a)$$
$$g(y) = G(y)(y-f(a)) + g(f(a))$$

Setting y = f(x) and h(x) = g(f(x)) yields

$$g(f(x)) = G(f(x))(f(x)-f(a)) + g(f(a))$$

or

$$h(x) = G(f(x))(F(x)(x-a) + f(a) - f(a)) + h(a)$$

or

$$h(x) = G(f(x))F(x)(x-a) + h(a)$$

Set H(x) = G(f(x))F(x). Since f, F and G are continuous, then so is H and we have

$$h(x) = H(x)(x-a) + h(a)$$

Thus, by Theorem 4.2, h(x) is differentiable and $h'(a) = H(a)$. This gives

$$(g\circ f)'(a) = G(f(a))F(a) = g'(f(a))f'(a).$$

Homework: page 93, #2, 4, 5
In Class: #1, 7, 8

4.12 – Rolle's Theorem: Suppose $a\neq b$, f differentiable on (a,b) and continuous on [a,b].
Then, $f(a) = f(b) \Rightarrow \exists\, c\in(a,b) \ni f'(c) = 0$.

Pf: By the extreme value theorem, there exist M and m so that $m \le f(x) \le M$.
If m = M, then $f'(x) = 0$ always.
If not, then f(x) is not constant on [a,b] and so there is a $c\in(a,b)$ where either f(c) = m or f(c) = M.
Wolog, assume f(c) = m. Then, $f(c+h) - f(c) \ge 0$, for all h such that $c+h\in(a,b)$.
However, for c+h < c, the left-hand derivative yields $f'(c) \le 0$ and for c+h > c, the right-hand derivative yields $f'(c) \ge 0$. Therefore, $f'(c) = 0$.

4.15 – Mean Value Theorem: Suppose $a\neq b$, f and g differentiable on (a,b) and continuous on [a,b].
Then $\exists\, c\in(a,b) \ni$

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

In particular, if g(x) = x, then

$$f'(c)(b - a) = f(b) - f(a).$$

Pf: Consider $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$.
Notice h(a) = h(b) and $h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a))$.
Hence, by Rolle's Theorem, $\exists\, c\in(a,b) \ni h'(c) = 0$ which yields the result.

4.17 – Bernoulli's Inequality: Suppose $\alpha>0$, $\delta \ge -1$. Then,

$$0 < \alpha < 1 \Rightarrow (1+\delta)^{\alpha} \le 1 + \alpha\delta$$
$$1 \le \alpha \Rightarrow (1+\delta)^{\alpha} \ge 1 + \alpha\delta$$

Pf: Consider $1 \le \alpha$ and $f(x) = x^{\alpha}$. Apply the MVT to f(x) on the interval from 1 to $1+\delta$ yields c such that

$$\alpha c^{\alpha-1}(1+\delta - 1) = f(1+\delta) - f(\delta)$$

or

$$f(\delta) + \alpha c^{\alpha-1}\delta = f(1+\delta)$$

$\delta > 0$ implies $c > 1$ and $\alpha-1 > 0$ yields $c^{\alpha-1} > 1$. Thus $\alpha c^{\alpha-1}\delta \ge \alpha\delta$ or $\delta c^{\alpha-1} \ge \delta$
$\delta < 0$ implies $c < 1$ and $\alpha-1 > 0$ yields $c^{\alpha-1} < 1$. Thus $\alpha c^{\alpha-1}\delta \le \alpha\delta$ or $\delta c^{\alpha-1} \ge \delta$

4.18 – L'Hopital's Rule: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ implies $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \lim_{x\to a}\dfrac{f'(x)}{g'(x)}$.

Pf: Uses Generalized Mean Value Theorem

Homework: page 100, #1, 4, 6, 10
In Class: #2, 5

4.24 – Monotonicity and Derivatives: Suppose f continuous on [a,b] and differentiable on (a,b). Then:

- $f'(x) > 0$ for all $x\in(a,b) \Rightarrow f(x)$ is strictly increasing for all $x\in(a,b)$
- $f'(x) < 0$ for all $x\in(a,b) \Rightarrow f(x)$ is strictly decreasing for all $x\in(a,b)$
- $f'(x) = 0$ for all $x\in(a,b) \Rightarrow f(x)$ is constant for all $x\in(a,b)$

Pf: Suppose $a \le x < w \le b$. The MVT applied to the interval (x,w) yields $c\in(x,w)$ such that

$$f(w) - f(x) = f'(c)(w - x).$$

$f'(c) > 0 \Rightarrow f(w) > f(x) \Rightarrow$ strictly increasing
$f'(c) < 0 \Rightarrow f(w) < f(x) \Rightarrow$ strictly decreasing
$f'(c) = 0 \Rightarrow f(w) = f(a)$, using $x = a \Rightarrow$ contantly equal to f(a) on the entire interval

4.25 – Families of Functions with Common Derivatives: Suppose f and g are continuous on [a,b] and differentiable on (a,b) such that $f'(x) = g'(x)$ for all $x\in(a,b)$. Then, $f(x) = g(x) + C$.

Pf: Let $h(x) = f(x) - g(x)$. Then, $h'(x) = 0$ for all $x\in(a,b)$. Apply 4.24 to obtain $h(x) = C$.

4.26 – 1-1 and monotonicity: f 1-1 and continuous implies f is strictly monotone.

Pf: Suppose f is a 1-1 function and continuous on some non-degenerate interval where $a < b$.
The one-to-oneness of f implies $f(a) < f(b)$ or $f(a) > f(b)$.
WOLOG, assume $f(a) < f(b)$.
For any point c such that $a < c < b$, if f is NOT strictly monotone, then either
$f(c) < f(a) < f(b)$ or
$f(a) < f(b) < f(c)$.
In the first case, applying the IVT yields a $w\in(c,b)$ with $f(w) = f(a)$ which contradicts 1-1.
In the second case, applying the IVT yields a $w\in(a,c)$ with $f(w) = f(b)$ which contradicts 1.1.
Therefore, f is strictly monotone.

4.26.1 – 1-1 and the monotonicity of inverses: f 1-1 and continuous implies f^{-1} is also continuous and strictly monotone.

4.27 – Inverse Function Theorem: Suppose f is 1-1 and continuous on an open interval I.
If $a\in f(I)$ and $f'(f^{-1}(a))$ exists and is nonzero, then f^{-1} is differentiable and

$$\left(f^{-1}\right)'(a) = \frac{1}{f'(f^{-1}(a))}$$

Pf: Theorem 4.26 implies f is strictly monotone and f^{-1} exists and is strictly monotone.
WOLOG assume f and f^{-1} are strictly decreasing.
Since I is open, then $f^{-1}(a)\in(c,d) \subseteq I$.
Since f is strictly decreasing, $f(d) < f(f^{-1}(a)) < f(c)$ and for small h, $f(d) < a + h < f(c)$.
Therefore, $d < f^{-1}(a+h) < c$ and so $f^{-1}(a+h)$ exists.
Thus, if $x = f^{-1}(a+h)$ and $x_0 = f^{-1}(a)$

$$f(x) - f(x_0) = f(f^{-1}(a+h)) - f(f^{-1}(a)) = a + h - a = h$$

Using the continuity of f^{-1}, we obtain

$$\frac{f^{-1}(a+h) - f^{-1}(a)}{h} = \frac{x - x_0}{f(x) - f(x_0)} = 1\Big/\frac{f(x) - f(x_0)}{x - x_0},$$

$$\lim_{h\to 0}\frac{f^{-1}(a+h) - f^{-1}(a)}{h} = 1\Big/\lim_{h\to 0}\frac{f(x) - f(x_0)}{x - x_0} = 1\Big/f'(f^{-1}(a)).$$

Homework: page 105, #4, 5
In Class: #1, 2, 3