Advanced Calculus Notes
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Based upon "An Introduction to Analysis", Wade, 3rd edition

CHAPTER 3

3.1 – Limit

3.4 – Equivalent Limits

3.6 – Sequential Characterization of Limits: $f(x)\to L$ for $x\to a \Leftrightarrow f(x_n)\to L\ \forall\ \{x_n\}\to a$ as $n\to\infty$.

Pf: Assume $f(x)\to L$ and let $\{x_n\}$ be any sequence that converges to a and let $\varepsilon > 0$.
By definition, there is a $\delta > 0$ where $0 < | x - a | < \delta$ implies $| f(x) - L | < \varepsilon$.
Also, by the definition of convergence for sequences, $\{x_n\}\to a$ implies that for $\varepsilon_1 = \delta > 0$, eventually

$$| x_n - a | < \delta.$$

For these $\{x_n\}$, $| f(x_n) - L | < \varepsilon$.

Conversely, assume $f(x_n)\to L\ \forall\ \{x_n\}\to a$ as $n\to\infty$.
By contradiction, assume there is some $\varepsilon_1 > 0$ where the definition of limit does not hold.
Therefore, there is an x such that $0 < | x - a | < \delta$ but $| f(x) - L | > \varepsilon_1$.
Using this ε_1, progressively, for various $\delta_n = 1/n$, pick an x_n where $| x_n - a | < 1/n$ but $| f(x_n) - L | > \varepsilon_1$.
Then, $\{x_n\}\to a$ and the hypothesis implies $f(x_n)\to L$, which is a contradiction.

3.8 – Combination of limits

3.9 – Squeeze Theorem

HOMEWORK: pg 62 #3, 7, 9
In Class: #1, 2, 8

Graduate Assignment:

- Read Section 9.1 generalizing sequential limits to $\Re^n$.
- Work and turn in Problems #2, 3, 5 and 8 on page 262.
- Read Section 9.2 generalizing functional limits to f: $\Re^n \to \Re$
- Work and turn in Problems #2, 3 and 4 on page 270

3.12 – Left Hand and Right Hand Limits

3.14 – Limits and One-sided Limits Theorem: $f(x)\to L$ for $x\to a \Leftrightarrow L$ is both the left-hand and right-hand limit

Pf: If the limit exists, then easily the conditions for left-hand and right-hand limits are both met.
Conversely, for any $\varepsilon>0$, there exist δ_1 and δ_2 such that the left-hand and right-hand limits exist.
Choosing $\delta = \min\{\delta_1, \delta_2\}$ makes the limit work.

Limits at infinity and limits of infinity - page 67

HOMEWORK: pg 68 #2, 4, 6

In Class: #1, 3, 5, 10

3.19 – Continuity

3.20 – Formula for Continuity: f is continuous ⇔ $f(x) \to f(a)$ as $x \to a$.

3.21 – Continuity and Sequential Continuity: f is continuous at a ⇔ $f(x_n) \to f(a)$ for all $x_n \to a$.

3.22 – Combinations of Continuous Functions are Continuous

3.24 – Continuity of Compositions: If f has a limit L at a and if g is continuous at L, then

$$\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x)).$$

Pf: Let $\varepsilon>0$.
Appealing to the sequential characterization of limits Theorem 3.17, consider the sequence $\{x_n\}\to a$.
Since f has a limit L, then $y_n = f(x_n)\to L$ and so there is some N such that $n \geq N$ implies $| f(x_n) - L | < \delta$, or $| y_n - L | < \delta$.
Since g is continuous at L, there is a $\delta>0$ such that $0 < | y_n - L | < \delta$ implies $| g(y_n) - g(L) | < \varepsilon$
Hence, $| g(f(x_n)) - g(L) | < \varepsilon$, as desired.

3.26 – Extreme Value Theorem: If f is continuous on a closed and bounded interval, then f is bounded. Further, if so, there exist points x_1 and x_2 such that $f(x_1) = \inf f(x) = m$ and $f(x_2) = \sup f(x) = M$.

Pf: If f is not bounded, then there exists a sequence $\{x_n\}$ such that $|f(x_n)| > n$, for all n.
However, the sequence $\{x_n\}$ must be in the bounded interval.
The Bolzano-Weierstrass Theorem implies that $\{x_n\}$ has a convergent subsequence...call it x_{nk}...which converges to some value a.
Since the interval is closed, there exists endpoints c and d such that $c \leq x_{nk} \leq d$.
The Comparison Theorem implies that a also lies in the interval.
Since f is defined on the interval, then f(a) is a finite real number.
On the other hand, $| f(x_{nk}) | > x_{nk}$, and so again by the Comparison Theorem, $f(x_{nk})\to\infty = f(a)$, a contradiction.

Therefore, f is bounded and so m and M are finite numbers.

Now, by contradiction, suppose $f(x) < M$ for all x in the interval.
Then, $g(x) = 1 / (M - f(x))$ is continuous on the closed interval and so (by above) g(x) is bounded.
Hence, there is a $C>0$ such that $g(x) \leq C$, or $f(x) \leq M - 1/C$.
Taking the sup over all x in the interval yields $M \leq M - 1/C$, which is a contradiction.
Similarly, we can show the result for the inf.

3.28 – Sign Preserving Property: f continuous at a and $f(a) > 0$ implies there exist $\varepsilon >0$ and $\delta > 0$ such that

$$|x-a|<\delta \text{ implies } f(x) > \varepsilon.$$

Pf: f continuous implies for $\varepsilon = f(a)/2 >0$, there is a δ such that $|x-a|<\delta$ implies

$$| f(x) - f(a) | < \varepsilon = f(a)/2.$$

Solve...

3.29 – Intermediate Value Theorem: Suppose $f:[a,b]\to\mathbb{R}$ is continuous. For any K between f(a) and f(b), there is a c between a and b such that $f(c) = K$.

Pf:
Wolog, assume $f(a) < K < f(b)$.
Extend f to have domain $\mathbb{R}$ by setting $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$.

Consider $E = \{ x : a \leq x \leq b$ and $f(x) < K \}$. Note $a \varepsilon E$ and so E is nonempty and bounded.
By the Completeness Axiom, set $c = \sup E$.
Theorem 2.11 implies there is a sequence $\{x_n\}$ in E such that $\{x_n\} \rightarrow c$
Since $a \leq x_n \leq b$, then $a \leq c \leq b$.
By the continuity of f, $f(c) = \lim f(x_n) \leq K$.
If $f(c) < K$, then $K - f(x)$ is continuous with $K-f(c)>0$
By the Sign Preserving Property 3.28, there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$K - f(x) > \varepsilon > 0 \text{ for } | x - c | < \delta.$$

This implies $f(x) < K - \varepsilon$, contradicting $c = \sup E$.
Since K is neither f(a) nor f(b), then c cannot be either a or b and so $a < c < b$, as desired.

HOMEWORK: pg 78 #3, 4, 9
In Class: #1, 2, 6

3.35 – Uniform Continuity

3.38 – Cauchy and Uniform Continuity: If f is uniformly continuous, then $\{x_n\}$ Cauchy implies $f(x_n)$ is Cauchy.

Pf: $\{x_n\}$ Cauchy implies $| x_n - x_m | < \delta$ implies $| f(x_n) - f(x_m) | < \varepsilon$.

3.39 – Closed and Bounded Continuity is Uniform: f continuous on a closed and bounded interval implies f is uniformly continuous on that interval.

Pf: By contradiction, suppose f is continuous but not uniformly.
Then, there is an $\varepsilon_0>0$ and points x_n and y_n such that $| x_n - y_n | < 1/n$ implies $| f(x_n) - f(x_m) | > \varepsilon_0$.
By the Bolzano-Weierstrass Theorem, $\{x_n\}$ has a convergent subsequence $\{x_{nk}\}$.
By the Comparison Theorem, $\{x_{nk}\}$ converges to x inside the original interval.
Similarly $\{y_n\}$ has a convergent subsequence $\{y_{nj}\}$ converging to y inside the original interval.
So, as $k, j \rightarrow \infty$, $| f(x_{nk}) - f(y_{nj}) | > \varepsilon_0$ implies $| f(x) - f(y) | \geq \varepsilon_0$.
But $| x_n - y_n | < 1/n$ implies $x = y$ which implies $f(x) = f(y)$, a contradiction.

HOMEWORK: pg 83 #2, 5, 9
In Class: #1, 3