Advanced Calculus Notes
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Based upon "An Introduction to Analysis", Wade, 3rd edition

CHAPTER TWO

2.1 – Convergence: A sequence of real numbers $\{x_n\}$ is said to *converge* to x if for every $\varepsilon > 0$, there is a natural number N such that $n \geq N$ implies $| x_n – x | < \varepsilon$.

2.2 – $1/n \to 0$ as $n \to \infty$

Pf: Given any $\varepsilon > 0$, use Archimedian principle with the numbers 1 and $1/\varepsilon$ to get N so that $N > 1/\varepsilon$. Then, for $n \geq N > 1/\varepsilon$ we get $\varepsilon > 1/n$, as desired.

2.4 – Uniqueness of Limits: A sequence can have at most one limit.

Pf: Suppose $\{x_n\}$ converges to both x and y.
Then, using the definition twice, for any $\varepsilon > 0$, there are natural numbers N and M such that
$n \geq N$ implies $| x_n – x | < \varepsilon /2$, and
$n \geq M$ implies $| x_n – y | < \varepsilon /2$.
So, by the triangle inequality
$| x – y | \leq | x_n - x | + | x_n - y | < \varepsilon$.
Apply theorem 1.9 to get $x – y = 0$ or $x = y$.

2.5 - Subsequences.

2.6 – Convergence of Subsequences: If a given sequence converges to x, then all possible subsequences converge to x.

Pf: Use defn, noting that $n_k \geq k > N$.

2.7 – Bounded Sequence: $\{ x_n \}$ is *bounded above* if and only if there is an M such that $x_n \leq M$ for all n and is *bounded below* if and only if there is an m such that $x_n \geq m$ for all n. $\{ x_n \}$ is *bounded* if both hold.

2.8 – Every convergent sequence is bounded.

Pf: Suppose $\{x_n\}$ converges to x.
Using the definition, for any $\varepsilon > 0$, there is a natural number N such that $n > N$ implies $| x_n – x | < \varepsilon$.
In particular, for $\varepsilon = 1$, this means there is a natural number N_1 such that $n > N_1$ implies $| x_n – x | < 1$.
Therefore $|x_n| = | x_n – x + x | \leq | x_n – x | + | x | < 1 + | x |$ for all $n > N_1$.
So, $| x_n | \leq \max\{ |x_1|, |x_2|, |x_3|, \ldots , |x_{N_1}|, | x | + 1\}$.

HOMEWORK: pg 37 #1, 2, 5, 7.
Work in class:

- #3
- #4 - Use defn with ε/C in place of ε.
- #6
- #8

2.9 – Squeeze Theorem: Suppose $x_n \le w_n \le y_n$ for $n > N_0$ and $x_n \to a$ and $y_n \to a$. Then, $w_n \to a$.

Pf: For $\varepsilon > 0$, there exists $N_1 > 0$ such that $| x_n – x | < \varepsilon$ for all $n > N_1$, or

$$x - \varepsilon < x_n < x + \varepsilon.$$

Further, there exists $N_2 > 0$ such that $| y_n – x | < \varepsilon$ for all $n > N_2$, or

$$x - \varepsilon < y_n < x + \varepsilon.$$

Then, for $N = \max\{N_0,N_1,N_2\}$,

$$x - \varepsilon < x_n \le w_n \le y_n < x + \varepsilon.$$

2.11 – If sup E is finite, then there is a sequence $x_n \in E$ such that $x_n \to \sup E$. Similarly for inf E.

Pf: By the approximation property for Supremums, for each n=1, 2, 3, ... with $\varepsilon = 1/n$, there exists an element x_n where $\sup E - 1/n < x_n \le \sup E$.

By the Squeeze Theorem, $x_n \to \sup E$.

2.12 – Limits obey the normal rules relative to addition, scalar multiplication, multiplication and division. (Have students put the proofs on the board.)

2.17 – Comparison Theorem: $x_n \le y_n$ implies $x = \lim x_n \le \lim y_n = y$.

Pf: By contradiction, assume not and consider $\varepsilon = (x – y)/2$.

Notice, $x > y$ means $\varepsilon > 0$.

Then $x_n > x - \varepsilon = y + \varepsilon > y_n$, which is a contradiction.

HOMEWORK: pg 42 # 2, 4, 5, 10

Work in class:

- #1 - (c) uses exercise 4
- #3
- #7 - note $x_n \to x$ implies $x_{n+1} \to x$ as well
- #9 - (a) can use exercise 1.2.1(c)

2.19 – Monotone Convergence Theorem: If $\{x_n\}$ is monotone and bounded, then is has a finite limit. That is, an increasing sequence converges to its supremum and a decreasing sequence converges to its infimum.

Pf: Assuming the sequence is increasing, completeness axiom guarantees the existence of a supremum.

The approximation property for suprema implies that eventually some term of this increasing sequence must pass $\sup\{x_n\} – \varepsilon$.

Monotonicity implies all subsequent terms remain close to $\sup \{x_n\}$.

2.23 – Nested Interval Property: The intersection of a nested sequence of non-empty closed bounded intervals is nonempty. Further, if the length of these intervals approaches zero, then this intersection contains exactly one number.

Pf: Use the Monotone Convergence Theorem on the (increasing) sequence of lower endpoints.

Since these are bounded, then they converge to some a.

Similarly, the (decreasing) sequence of upper endpoints converge to some b.

Easily $a \le b$.

Further, if the length approaches zero, then a=b.

2.26 – Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.

Pf: Let $\{x_n\}$ be a bounded sequence.
Then, there exists an interval such that $a \leq x_n \leq b$ for all n.
Denote $I_0 = [a,b]$.
Divide this into two halves.
At least one must contain infinitely many terms in $\{x_n\}$.
Denote this half by I_1 and pick any element of the sequence x_{n1} which lives in I_1.
Notice, $|I_1| = |I_0| = (b-a)/2$.
Continuing this process, choose intervals $I_2, I_3, I_4, I_5, \ldots I_n, \ldots$ and natural numbers $n_2, n_3, n_4, n_{25} \ldots n_n, \ldots$ such that elements $x_{n2}, x_{n3}, x_{n4}, x_{n5}, \ldots x_{nn}, \ldots$ belong to the appropriate interval and $|I_n| = (b-a)/2^n$.
By The Nested Interval Property (Theorem 2.23), there is an x in the intersection of all of these intervals where $|x-x_{nk}| < (b-a)/2^k$ which implies the subsequence converges to x, as desired.

HOMEWORK: pg 47 #2, 5, 10

Work in class:

- #1 - note, you only have to prove there exists a convergent subsequence. Not actually find it.
- #4 - use exercise 1.2.4(a)
- #6 - use exercise 1.2.4(b)
- #8 - prove $\{x_n\}$ is monotone
- #9 - (a) use exercise 1.1.5(c)

2.28 – $\{x_n\}$ convergent implies $\{x_n\}$ is Cauchy.

Pf: $| x_n-x_m | \leq | x_n-x | + | x-x_m |$

2.29 – For real sequences, $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ is convergent.

Pf: Remark 2.28 gives the proof one direction.
So, assume $\{x_n\}$ is Cauchy.
Since the x_n's get closer to each other, then there exists N such that for m>N,

$$|x_M - x_m| < 1 \text{ or } |x_m| < |x_M| + 1.$$

The other first M terms also are bounded and so the sequence must be bounded.
By the Bolzano-Weierstrass Theorem, $\{x_n\}$ has a convergent subseqeunce, say $x_{nk} \rightarrow a$.
So, by the defn of convergence, there is N_1 such that $n_k > N_1$ implies $|x_{nk} - a| < \varepsilon/2$.
Further, the original sequence Cauchy implies for $n > N_2$ yields $|x_n - x_m| < \varepsilon/2$.
So, for n sufficiently large, $|x_n - a| \leq |x_n - x_{nk}| + |x_{nk} - a| < \varepsilon$, as desired.

HOMEWORK: pg 50 #1, 3, 6

Skip section 2.5