Advanced Calculus Notes
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Based upon "An Introduction to Analysis", Wade, 3rd edition

CHAPTER ONE

P1 - Field Axioms - pg 2

P2 - Order Axioms - pg 4

1.7 – Triangle Inequalities:

- $| a + b | \leq | a | + | b |$
- $| a – b | \geq | a | – | b |$
- $| \; | a – b | \; | \leq | a – b |$

1.9 – Results regarding closeness and smallness:

- $x < y + \varepsilon, \ \forall \varepsilon > 0 \Leftrightarrow x \leq y$. Pf: Suppose also $x > y$ and let $\varepsilon = x – y$.
- $x > y – \varepsilon, \ \forall \varepsilon > 0 \Leftrightarrow x \geq y$. Pf: Rewrite as $-x < -y + \varepsilon$ and use first result.
- $| x | < \varepsilon, \ \forall \varepsilon > 0 \Leftrightarrow x = 0$. Pf: $0 – \varepsilon < x < 0 + \varepsilon$. Use both results above with $y = 0$.

HOMEWORK - pg 11 #1, 4, 6, 8.
Work in class:

- #3 - (b) consider cases $c = 0$ and $c > 0$
- #5 - (a) apply (6) to 1- a. (b) apply (6) to a - 1. (c) note $(\downarrow a - \downarrow b)^2 \geq 0$
- #10 - (a) note $|xy - ab| = |xy - xb + xb - ab|$ and $|x| < |a| + \varepsilon$

P3 - Well-Ordering Principle (WOP): Every nonempty subset of the natural numbers has a least element.

Note: The real numbers, integers and rational numbers have no least elements.

This basic axiom has no proof but accepting its validity allows us to prove many very important and reasonable results.

The Fundamental Theorem of Arithmetic: Every positive integer greater than one can be expressed uniquely as a product of primes, except for the arrangement of terms.

Pf: If n is prime, $n = 1 * n$ and we are done.
If n is composite, then n has a smallest positive divisor p_1 other than one and itself.
Since p_1 is a smallest divisor, it must also be prime. Indeed, suppose there is an integer q such that $1 < q < p_1$ and $q \mid p_1$. Then, $q \mid n$, which contradicts p_1 being the smallest positive divisor.
Therefore, $n = p_1 n_1$ where p_1 is prime and $n > n_1$.
Repeat this argument starting now with n_1 …
If n_1 is prime, then $n = p_1 n_1$ is a product of primes and we are done.
If n_1 is composite, then $n_1 = p_2 n_2$, where p_2 is a prime smallest nontrivial divisor of n_1 and $n_1 > n_2$.
Continuing the argument with n_2, ... yields a (decreasing) sequence $n > n_1 > n_2 > n_3 \ldots$ of natural numbers. Apply the Well-Ordering Principle to yield a least element, say p_m, which must then be prime.
Then $n = p_1 p_2. \ldots .p_m$ and we are done.

1.11 - Principle of Mathematical Induction: For a conditional proposition P(n) defined on the natural numbers, if P(1) is true and if P(k) => P(k+1), for all k≥1, then P(n) is true for all n≥1.

Pf: By contradiction, suppose the theorem is false and let E = {n : P(n) is false }.

Since we have supposed that the theorem is false, there must be some n_0 where $P(n_0)$ is false and so E is nonempty. By the WOP, E has a least element, say at x. Since P(1) is true, then x is not 1. Since x is an integer, then it has a predecessor x-1 which is also an integer and where P(x-1) is true. However, using the second hypothesis, then we must have P(x-1+1) = P(x) true, which is a contradiction.

Graduate Assignment: Write a paper on the uses of the Well-Ordering Principle. In particular, note its relationship with several other equivalent axioms. (Zorn's Lemma, Axiom of Choice) Be sure to prepare your paper so that an undergraduate student in this class would be able to read without too much work. Be ready to discuss your findings during a classroom presentation.

1.15 - Binomial Theorem - $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Pf: $(a+b)^1 = \sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k}$ is trivially true.

Assume $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ and consider

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} \\
&= \sum_{j=1}^{n+1} \binom{n}{j-1} a^j b^{n+1-j} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} \\
&= \sum_{i=0}^{n+1} \binom{n+1}{i} a^i b^{n+1-i}
\end{aligned}$$

Therefore, by the PMI, the formula holds for all natural numbers.

HOMEWORK - pg 17 #1, 2, 6, 7.

Work in class:

- #3
- #4 - See exercise 1.1.5
- #8 - (a) Show that $n^2 + 3n$ cannot be the square of an integer when n>1 (b) Rational only if n=9.

1.20 - Approximation Property for Suprema: Let E be a set with supremum sup E and let ε>0. Then there is a point a∈E such that sup E - ε < a ≤ sup E.

Pf: By contradiction, suppose the theorem is false.

Then, there is some $\varepsilon_0 > 0$ such that no element of E lies in the interval (sup E - ε_0, supE] and so all elements a∈E satisfy a ≤ sup E - ε_0. Hence, sup E - ε_0 is also an upper bound for the set E.

Since the supremum is the least upper bound, then sup E ≤ sup E - ε_0 or $\varepsilon_0 \le 0$, which is a contradiction.

P4 - The Completeness Axiom: If E is a nonempty subset of the real numbers which is bounded above, then E has a finite supremum.

1.22 - Archimedian Principle: Given positive real numbers a and b, there is an integer n such that b < n a.

Pf: See page 19

1.24 - Density of the Rationals: For any real numbers a and b where $a < b$, there is a rational q such that

$$a < q < b.$$

Pf: Since $b - a > 0$, use the Archimedian Principle to obtain a natural number n such that $n(b-a) > 1$.
Then, $b-a > 1/n$ or $-(b-a) < -1/n$.
<u>Case 1</u>: $b>0$.
Consider the set $E = \{ k : b \leq k/n \}$. By the Archimedian Principle applied to b and 1/n, E is nonempty.
By the WOP, E has a least element, say k_0.
Let $m = k_0 - 1$ and $q = m/n$.
Since k_0 is a minimal element of E, then m is not in E...hence, $b > m/n$.
Further, $a = b - (b-a) < k_0/n - 1/n = m/n$.
Thus, we obtain the desired result $a < m/n < b$.
<u>Case 2</u>: $b \leq 0$
By the Archimedian Principle applied to -b and 1, choose a natural number k such that $-b < k \cdot 1$ or $0 < k+b$.
By Case 1, applied to k+a and k+b (which is positive), there is a rational r such that $k+a < r < k+b$.
Subtracting the integer k yields $a < r-k < b$, where r-k is indeed a rational.

1.29 – The Monotone Property: Suppose $A \subseteq B$ are nonempty real sets. Then:

- If sup B exists, then $\sup A \leq \sup B$.
- If inf B exists, then $\inf B \leq \inf A$.

Pf: See page 22

HOMEWORK: pg 23 #1, 5, 6.
Work in class:

- #3
- #4
- #7

Review definitions and results relating to functions in section 1.4

Mean Value Theorem: From Calculus I, if f is continuous on [a,b] and differentiable on (a,b), then there exists a $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.
Pf: Coming later in chapter 4.

1.32 – Sufficient Condition for a function to be 1-1: If f is differentiable on a open interval (a,b) and $f'(x) \neq 0$ for all x in (a,b), then f is 1-1 on (a,b).

Pf: By contradiction, suppose f is not 1-1 on I. Then, there exists $c, d \in (a,b)$ such that $f(c) = f(d)$.
So, by the MVT, there is a $\xi \in (c,d)$ such that $0 = [f(d) - f(c)] / (d - c) = f'(\xi) \neq 0$, which is a contradiction.

HOMEWORK: pg 32 #2, 4, 10.
Work in class:

- #5