Dr. John Travis, Mississippi College - Notes on Differential Operators

Defn: An *nth order linear* DE has the form

(L) $$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + ... + a_1(t)y' + a_0(t)y = g(t),$$

where the functions $a_k(t)$ and $g(t)$ are continuous functions on some interval (a,b).
We call g(t) the *forcing term*.

If g(t) = 0, then the DE is called *homogeneous* (LH); else *non-homogeneous* (LN).

If n = 1, then (L) is know as *First Order Linear* and can be solved using an integrating factor.

If $a_k(t) = a_k$ = constant for all k, the DE has *constant coefficients* and is denoted (LHC).

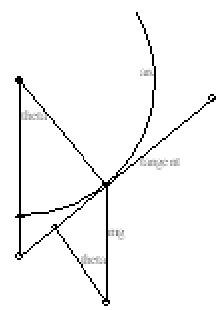
Remark: For a DE to be linear, the following must be noted:

(1) All the unknown function terms are raised only to the first power.
(2) There are no cross terms involving the unknown function or its derivatives.

Remark: We will assume that $a_n(t)$ is nonzero for all t values in a given interval (a,b). Hence, the DE is always nth order throughout (a,b).

Application: Pendulums

Let a mass m be suspended by a (nonflexible) rod of length r. Let θ be the angle the rod is from vertical and let x = r θ be the arc length from vertical. Once θ is known, the precise location of the mass is also known.



The *amplitude* is the maximum θ and the *period* is the time required for the pendulum to go through a complete cycle.

Gravity will not affect any motion of the pendulum in the middle (at *equilibrium*) but will affect it otherwise. For our purposes, we consider the motion of the mass in the direction of its tangent with gravitational force component -mg sin(θ) where the negative sign indicates a *restoring force* always directed toward the equilibrium.

- Assume the rod is massless and the mass m to be concentrated at one point. Negle friction in the hinge and fluid resistance (ie. the pendulum is in a vacuum). With these assumptions, m x" = ma = F = -mg sin(θ), or
x" + g sin(θ) = 0.

But, arc length x = r θ implies x" = r θ" which yields the nonlinear model r θ" + g sin(θ) = 0. We can "linearize" this by using the Maclaurin expansion of $\sin(\theta) = \theta - \theta^3/3! + ...$ or approximately we replace the sin(θ) term with θ provided the mass moves through a reasonably small angle θ. This yields the linear approximate model $r\,\theta'' + g\,\theta = 0$.

- Assume now that the pendulum is damped (air or fluid resistance).
This yields $mx'' = -mg\sin(\theta) - bx'$, or as above we obtain $mr\,\theta'' + br\,\theta' + mg\,\theta = 0$.

- Assume we add an external force $f(t)$.
This yields $mr\,\theta'' + br\,\theta' + mg\,\theta = f(t)$.

Result: (Superposition Principle) If (LH) has solutions $y_1(t)$ and $y_2(t)$, then $c\,y_1(t) + d\,y_2(t)$ also solves (LH).

Remark: We would like to determine the general solution to a nth order D.E. To do so, we need to find the most general formula which, by a proper choice of constants only, gives <u>any</u> solution of the DE. By the superposition principle, we see that any linear combination of two solutions yields a solution which contains both as special cases.

Defn: The set of n functions $f_1(t), f_2(t), \ldots f_n(t)$ is *linear dependent* on a given interval if and only if constants $b_1, b_2, \ldots, b_n$ exist (not all zero) such that

$$b_1 f_1(t) + b_2 f_2(t) + \ldots b_n f_n(t) = 0,$$

for all t in the interval. If no such constants exist, the functions are *linearly independent*.

Special Case: Two functions are linearly dependent on an interval iff one is a constant multiple of the other.

Result: (LH) has n linearly independent solutions over (a, b) provided the coefficients are continuous over (a, b) and $a_n(x)$ is nonzero over (a, b).

Defn: Let $y_1(t), \ldots, y_n(t)$ be linearly independent solutions of (LH) over (a,b).
Then, $y_1(t), \ldots, y_n(t)$ are called a *fundamental set of solutions* and the *general solution* of (LH) is given by

$$y(t) = c_1\, y_1(t) + \ldots + c_n\, y_n(t),$$

where the values c_k are arbitrary constants.

Solving LHC: Assume a solution of the form $y(t) = e^{\lambda t}$, where λ is a unknown constant. Substitute into (LHC) to obtain after simpification

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0,$$

which is an algebraic equation known as the *auxiliary equation*. Hence, values of λ which make the auxiliary equation true give solutions for (LHC) using $y=e^{\lambda t}$.

Result 1: If r and s are distinct numbers, then the functions e^{rt} and e^{st} are linearly independent. (A similar result holds for an arbitrary number of distinct real numbers.)

Result 2: For a given value of λ, the functions $e^{\lambda t}$, $te^{\lambda t}$, $t^2e^{\lambda t}$, $t^3e^{\lambda t}$, ... are linearly independent.

Result 3: (Euler's Identity) For the pure imaginary number $i\beta$, $e^{i\beta} = \cos(\beta) + i\sin(\beta) = \text{cis}(\beta)$.
Pf: Using Maclaurin,
$e^{i\beta} = 1 + (i\beta) + (i\beta)^2/2! + (i\beta)^3/3! + (i\beta)^4/4! + (i\beta)^5/5! + ...$
$= \{1 + (i\beta)^2/2! + (i\beta)^4/4! + ...\} + \{(i\beta) + (i\beta)^3/3! + (i\beta)^5/5! + ...\}$
$= \{1 - \beta^2/2! + \beta^4/4! - ...\} + i*\{\beta - \beta^3/3! + \beta^5/5! - ...\} = \cos(\beta) + i\sin(\beta)$.

Result 4: $e^{\alpha x}\cos(\beta)$ and $e^{\alpha x}\sin(\beta)$ are linearly independent, provided β is nonzero.

Result 5: Complex roots to the auxiliary equation must occur in conjugate pairs of the form $\alpha + \beta i$ and $\alpha - \beta i$.

Defn: A *function* generallycorresponds a number to a number.
A *functional* corresponds a function to a number. (Like the definite integral.)
An *operator* corresponds a function to a function. (Like the indefinite integral or the derivative.)

Defn: Define the kth derivative operator D^k by the formula $D^k\{f\} = f^{(k)}(t)$.
Define the *identity* operator I by the formula $I\{f\} = f(t)$.

Remark: We can restate (L) using the differential operator D^k in the form

$$L\{y\} = a_n(t)D^n y + a_{n-1}(t)D^{n-1}y + ... + a_1(t)Dy + a_0(t)Iy = g(t),$$

or in the operator form

$$L\{y\} = g(x).$$

Then, a solution y(t) to (L) is also a solution to the operator equation $L\{y\} = g(t)$.

Special Case: (LHC). Then L{y} = 0 becomes

$$a_n D^n y + a_{n-1} D^{n-1} y + \ldots + a_1 Dy + a_0 Iy = 0.$$

By using the linearity of the differential operator, this becomes

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0 I)y = 0.$$

Notice, if we define the polynomial $P(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$, this becomes

$$P(D)y = 0.$$

Results on Differential Operators:

1. $D^k[e^{\lambda t}] = \lambda^k e^{\lambda t}$.

2. $P(D)e^{\lambda t} = e^{\lambda t} P(\lambda)$, by reapplying (1) on each term of P(D).
Therefore, roots λ of $P(\lambda)$ give solutions to P(D)y=0.

3. $(D-\lambda)[e^{\lambda t} y] = e^{\lambda t} Dy$, by using the product rule and linearity.
Hence, $(D-\lambda)[e^{\lambda t}] = e^{\lambda t} D[1] = 0$.

4. $(D-\lambda)^k[e^{\lambda t} y] = e^{\lambda t} D^k y$, by recursively applying (3).

5. $(D-\lambda)^k[t^r e^{\lambda t}] = 0$, for r = 0,1,...,k-1, by applying (4) with $y = t^r$, noting $D^k t^r = 0$ since $r < k$.

6. To solve (LHC):
   Write (LHC) in the form P(D)y=0.
   Factor $P(t) = Q(t)(t-\lambda)^k$, where Q is a polynomial of degree n-k.
   $(D-\lambda)^k y = 0$ precisely when $y = t^r e^{\lambda t}$, for r=0,1,...,k-1, using (5).
   So, $P(D)y = Q(D)(D-\lambda)^k y = 0$. Therefore, $y = t^r e^{\lambda t}$ are solutions for r=0,1,...,k-1.

CASES: (1) Real, distinct roots
(2) Real, repeated roots
(3) Complex, distinct roots
(4) Complex, repeated roots.

Solutions to LHC: Determine the roots of the auxiliary equation.
CASE 1: Distinct real roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then, the general solution is

$$y(t) = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t) + \ldots c_n \exp(\lambda_n t).$$

CASE 2: The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers but some are repeated. Then, the general solution to the DE is given by the sum of terms of the form

$$c_1 \exp(\lambda t) + c_2 t \exp(\lambda t) + \ldots c_k t^{k-1} \exp(\lambda t),$$

where the root λ is repeated k times.

CASE 3: The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ include one complex pair $\alpha + \beta i$ and $\alpha - \beta i$.
Then, $y_1 = e^{(\alpha + \beta i)t}$ and $y_2 = e^{(\alpha - \beta i)t}$ solve (LHC).

By Euler's Identity, $y_1 = e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))$ and $y_2 = e^{\alpha t}(\cos(\beta t) - i \sin(\beta t))$.

By the superposition principle, both $y_3 = (y_1 + y_2)/2$ and $y_4 = (y_1 - y_2)/(2i)$ solve (LHC).
But, $y_3 = e^{\alpha t} \cos(\beta t)$ and $y_4 = e^{at} \sin(\beta t)$.

Therefore, $c_1 y_3 + c_2 y_4$ solves (LHC) and the general solution is
$y(x) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$ + other terms.

CASE 4: The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. include a complex pair $\alpha + \beta i$ and $\alpha - \beta i$. repeated k times.
Then, the general solution to (LHC) includes

$$y_k(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t)) + t\, e^{\alpha t}(c_3 \cos(\beta t) + c_4 \sin(\beta t)) \\ + t^2 e^{\alpha t}(c_5 \cos(\beta t) + c_6 \sin(\beta t)) + \ldots + t^{k-1} e^{\alpha t}(c_{2n-1} \cos(\beta t) + c_{2n} \sin(\beta t)).$$

Application: Automobile Suspension Systems Revisited
- Now assume the mass m is the portion of the car's mass supported by one tire. The same equation developed earlier still holds.
- Determine the spring and shock absorber constants to give a good, smooth, safe ride.
- Model 1: No spring or shock. No good.
- Model 2: Using a spring but no shock. Oscillatory results or *Simple Harmonic motion.*
- Model 3: Both spring and shock used. *Damped* results.
- Model 4: Add an external force. Since we have not yet solved (LNC) yet, we must wait till later

HOMEWORK: Work the following problems:

1. $y'' - 3y' = 0$

2. $y'' - 3y' - 10y = 0$

3. $y''' - y = 0$

4. For differential equation $y'' - \mu y' + \omega^2 y = 0$, for fixed ω, determine bifurcation values for μ.

Solving the Non-homogeneous Problem

Generalized Superposition Principle: If $y_c(t)$ is the general solution of (LHC) and $y_p(t)$ is any solution for (LNC), then the general solution for (LNC) is $y(t) = y_c(t) + y_p(t)$.

Special Differential Operators

- $D^k [t^j] = 0$, provided $0 \leq j < k$.
- $(D-\lambda)^k [t^j e^{\lambda t}] = 0$, provided $0 \leq j < k$.
- $(D^2 + \beta^2)^k [t^j \sin(\beta t)] = 0$, provided $0 \leq j < k$.
- $(D^2 + \beta^2)^k [t^j \cos(\beta t)] = 0$, provided $0 \leq j < k$.
- $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^k [t^j e^{\alpha t} \sin(\beta t)]$, provided $0 \leq j < k$.
- $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^k [t^j e^{\alpha t} \cos(\beta t)]$, provided $0 \leq j < k$.

To solve LNC using the method of Undetermined Coefficients:

1. Solve the associated homogeneous DE to obtain $y_c(t)$.
2. By looking at the forcing term, determine an appropriate differential operator and apply to both sides in order to create a higher order homogeneous differential equation.
3. Solving this new homogeneous DE, take all terms not already part of $y_c(t)$ above as candidates for $y_p(t)$ with arbitrary constants as coefficients which will be "determined".
4. Plug $y_p(t)$ above into original DE and equate like terms. This should yield a system of linear equations which can be solved for these now "determined" coefficients.
5. Write the general solution to (LNC) as $y(t) = y_c(t) + y_p(t)$.

Homework: Solve some problems in textbook, chapter 4, sections 1, 2 and 3.