Laplace Transforms

Math 352 Lecture Notes

Dr. John Travis

Mississippi College

Remark: In solving (LNC), assuming that the forcing term g(t) is "nice" allows one to use the method of Undetermined Coefficients. However, in real-life applications, forcing functions may not present themselves as one of the canonical forms already discussed and may even exhibit a number of discontinuities. We would like to develop the mathematics for dealing with these problems.

First Idea: Determine each of the points at which g(t) is discontinuous. If this set is discrete, it breaks up the problem in to a sequence of initial value problems on each interval for which the ending value for a given interval is the initial value for the next interval. This can be successfully implemented but if the number of intervals is great, it requires us to solve a new DE for each one.

Defn: The Laplace transform **L** is an operator corresponding a function f(t) to a function F(s) given by

$$F(s) = \mathbf{L}\{ f(t) \} = \int_0^\infty e^{-st} f(t)\,dt ,$$

provided the infinite integral exists.

Results:

1. $\mathbf{L}\{ \alpha \} = \alpha / s$, for α any constant.
2. $\mathbf{L}\{ t \} = 1/s^2$.
3. $\mathbf{L}\{ t^n \} = 1/s^{n+1}$. (Use integration by parts and induction. A later result could prove this more easily.)
4. $\mathbf{L}\{ e^{\alpha t} \} = 1/(s-\alpha)$.
5. $\mathbf{L}\{ \sin(\beta t) \} = \beta/(s^2+\beta^2)$
6. $\mathbf{L}\{ \cos(\beta t) \} = \beta s/(s^2+\beta^2)$

Defn: The Unit Step function $U(t) = 0$, if $t < 0$ and $U(t) = 1$, if $t \geq 0$. This function is often utilized in a shifted and/or reflected mode. Indeed, consider $U(-t)$, $U(t-\alpha)$ and $U(\alpha-t)$.

Remark: One may construct a piecewise function using a single formula using the unit step function by successively piecing terms together. Indeed, if

$f(t) = A$, for $t \leq \alpha$, and

$f(t) = B$, for $\alpha < t \leq \beta$, and

$f(t) = C$ for $t > \beta$,

then

$$f(t) = A + U(t - \alpha)\,(B - A) + U(t - \beta)\,(C - B).$$

Defn: The Dirac delta function is given by $\delta(t) = 0$, for $t \neq 0$ and $\delta(t) = \infty$ for $t = 0$. The delta function allows one to model impulses in forcing functions.

Result: $\mathbf{L}\{\, \delta(t - \alpha) \,\} = e^{-\alpha s}$.

Linearity properties of the Laplace Transform:

1. $\mathbf{L}\{\, c\, f(t) \,\} = c\, \mathbf{L}\{\, f(t) \,\}$
2. $\mathbf{L}\{\, f(t) + g(t) \,\} = \mathbf{L}\{\, f(t) \,\} + \mathbf{L}\{\, g(t) \,\}$.
3. $\mathbf{L}\{\, c\, f(t) + d\, g(t) \,\} = c\, \mathbf{L}\{\, f(t) \,\} + d\, \mathbf{L}\{\, g(t) \,\}$.

Defn: If $\mathbf{L}\{\, f(t) \,\} = F(s)$, define the inverse Laplace transform of $F(s)$ to be $f(t)$. That is,

$$\mathbf{L}^{-1}\{\, F(s) \,\} = f(t).$$

Result: The inverse Laplace transform is linear. That is,

$$\mathbf{L}^{-1}\{\, \alpha\, F(s) + \beta\, G(s) \,\} = \alpha\, \mathbf{L}^{-1}\{\, F(s) \,\} + \beta\, \mathbf{L}^{-1}\{\, G(s) \,\} = \alpha\, f(t) + \beta\, g(t).$$

Defn: The function $f(t)$ is said to be of *exponential order* if there exists positive constants M and c and a number $T > 0$ such that $|f(t)| \leq M\, e^{c\,t}$, for all $t > T$.

Existence Results: If $f(t)$ is piecewise continuous for $t \geq 0$ and of exponential order, then the

Laplace transform exists for $s > c$. $F(s)$ has an inverse transform of exponential order provided, as s approaches infinity, $\lim F(s) = 0$.

Remark: The result above gives us a necessary condition for a given function in s to have an inverse transform which is of exponential order. However, it does not tell us how to find the function f(t). The way we determine f(t) is by reversing many of the rules already derived. In doing this, we will often need to use partial fractions to break up terms.

Theorem: Suppose f and g are continuous functions of exponential order for $t \geq 0$.

If $F(s) = G(s)$, then $f(t) = g(t)$.

First Translation Theorem: (Translation in transform space)

If α is any real number and $F(s) = \mathbf{L}\{ f(t) \}$, then

$$\mathbf{L}\{ e^{\alpha t} f(t) \} = \mathbf{L}\{ f(t) \}|_{s:=s-\alpha} = F(s - \alpha).$$

By reversing the this result, we also obtain

$$\mathbf{L}^{-1}\{ F(s - \alpha) \} = e^{\alpha t} f(t).$$

Result: If f(0) is the value of the function f(t) at t=0,

$$\mathbf{L}\{ f'(t) \} = s F(s) - f(0).$$

By reversing this result, if $f(0) = 0$ we also obtain

$$\mathbf{L}^{-1}\{ s F(s) \} = f'(t).$$

Transforms of Derivatives Theorem: In general, by reapplying the result above**,**

$$\mathbf{L}\{ f^{(n)}(t) \} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

In particular, this allows us to transform DE's into algebraic equations.

Result: $F'(s) = \mathbf{L}\{ -t f(t) \}$

Derivatives of Transforms Theorem: In general, by reapplying the result above,

$$F^{(n)}(s) = \mathbf{L}\{ (-t)^n f(t) \}.$$

By reversing this result, we also obtain

$$\mathbf{L}^{-1}\{ F^{(n)}(s) \} = (-t)^n f(t).$$

Remark: We plan on using the Laplace transform to solve (LN). To do so, we use the above results as follows:

1. Apply the Laplace Transform to both sides of (LN).
2. Apply Transforms of Derivatives Theorem to all derivative terms. In doing so, you may have some terms which require the Derivative of Transforms Theorem.
3. Solve the resulting algebraic/differential equation for Y(s).
4. Apply the Inverse Laplace Transform to both sides.
5. Solve for the solution y(t).

Notice: We are allowing the non-constant coefficient case here. For terms with variable coefficients, we will need to apply the Derivative of Transforms Theorem. This will unfortunately yield a differential equation in the variable s. However, if the coefficients aren't too bad, then this new DE will be much easier to solve.

Remark: We would like to solve (LN) when we have a piecewise continuous forcing term g(x). However, we need to express g(t) as one formula so that the DE will be just one equation valid for all values of time.

2nd Translation Theorem: (Translation in function space) $\mathbf{L}\{ f(t) \} = F(s)$ and $\alpha > 0$ implies

$$\mathbf{L}\{ f(t-\alpha)U(t-\alpha) \} = e^{-\alpha s} F(s), \text{ for } t > \alpha.$$

By reversing this result, we also obtain

$$\mathbf{L}^{-1}\{ e^{-\alpha s} F(s) \} = f(t-\alpha)\, U(t-\alpha),$$

which is a function zero till $t = \alpha$ and then $f(t-\alpha)$ after $t = \alpha$.

Remark: To apply this result, all functions involved need to be in the form $f(t - \alpha)$. To convert an arbitrary function of t to a function of $t - \alpha$, start with the highest order terms t^k and write them as $(t - \alpha)^k$. Take leftover terms and simplify them with the lower order terms. Continue this till all terms are in $t - \alpha$.

Defn: $f(t)$ is <u>periodic</u> of period τ if $f(t + \tau) = f(t)$, for all t.

Theorem: If $f(t)$ is periodic with period τ and of exponential order, then the Laplace transform of $F(t)$ can be expressed as a proper intergral over the interval $[0,\tau]$. See book for result.