CHAPTER TWO - SYSTEMS

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Predator-Prey Model

Let the population of Prey be denoted by R(t) (for rabbits) and the population of Predators by F(t) (for foxes)

Consider the following assumptions:

- In the absence of Predators, the Prey will grow unrestricted. Hence, R' must have a term of the form "aR"
- Predators eat Prey at a rate proportional to how often they interact. This can be modeled with a term of the form "-b R F".
- In the absence of Prey, Predators will die off. Hence, F' must have a term of the form "- cF".
- The growth rate of Predators is proportional to the number of Prey eaten by the Predator. This yields a term "dRF".

Gathering these assumptions yield a first order system:

$$R' = aR - bRF$$
$$F' = -cF + dRF$$

The system is *coupled* because the rates R' and F' both depend upon R and F.

The system is called *autonomous* since the independent variable t does not appear.

General Autonomous Systems of Differential Equations

Consider the set of DE's

$$x' = f(x,y)$$
$$y' = g(x,y)$$

Using vector notation, set

$$\mathbf{Y}(t) = (\ x(t), y(t)\ ),$$
$$\mathbf{F}(Y) = (\ f(x,y)\ ,\ g(x,y)\ ) \text{ and}$$
$$d\mathbf{Y}/dt = (\ dx/dt\ ,\ dy/dt\ ).$$

Then, the system can be written:

$$\mathbf{Y}' = \mathbf{F}(\mathbf{Y})$$

An initial condition for this systems is a pair of values, one for each unknown function. Such an initial condition **Y**(t_0) = (x(t_0), y(t_0)) can be specified to create an initial value problem for systems (SIVP).

This can be easily extended to cover more than two coupled DEs.

Terms:

- The *dimension* of the first-order system is the number of dependent variables.
- The system is called *linear* provided **F**(Y) has all dependent variables to at most the first power.
- The system is called *autonomous* if t does not explicitly appear.
- **Y**(t) = (x(t), y(t)) is a *solution* provided it satisifes the DE and any IC.
- A constant solution (in all equations) will be called an *equilibrium solution*.
- A coordinate system consisting only of the dependent variables is the *phase plane* for the DE.
- Graphing the solution parametrically in the phase plane yields a curve called the *solution curve*.
- Plotting many solution curves in the phase plane yields the *phase portrait*.

ASSUMPTION: Unless otherwise stated, we will assume in chapter 2 that all DEs are autonomous.

VECTOR FIELD and **DIRECTION FIELD**

Plot **F**(**Y**) for various values of **Y**. Vector field plots both direction and magnitude of **Y**' at each **Y** value while direction field only plots equal length directions at each **Y**.

PHASE PLANE: (Phase Portrait)

A Comparison of solutions by plotting (x(t), y(t)), much like the direction field. Using the direction field as a starting point, denote all equilibrium points by dots. Then, display a representative collection of solution curves as t increases.

Periodic Solutions: Often, solutions will repeat after an interval of time for these problems. They behave in a cyclic manner. If so, we say they are *periodic* and should satisfy, for some parameter p (called the period),

$$x(t+p) = x(t),$$
$$y(t+p) = y(t),$$

for all t.

HOMEWORK: page 144 #1-6, 7, 25

Defn: For the autonomous system of differential equations $\mathbf{Y}' = \mathbf{F}(\mathbf{Y})$, $\mathbf{Y}_0$ is an equilibrium point if $\mathbf{F}(\mathbf{Y}_0)=0$. The constant function $\mathbf{Y}(t) = \mathbf{Y}_0$ is an equilibrium solution.

Note, if $\mathbf{Y}' = 0$, then the direction is any direction but at no velocity. That is, we are stuck at the equilibria and can't get moving!

HOMEWORK: page 158 #8, 9-16, 17-20, 29

Graphing Solutions: We can express the solutions to a 2-dimensional system in several ways.

- Phase Plane - graphing solutions (x(t). y(t))
- Time Series - graphing (t. x(t)) and (t. y(t))
- 3D - graphing in 3 dimensions (t, x(t), y(t))

Theorem: Existence and Uniqueness of Solutions are guaranteed provided all partials are continuous - page 167

HOMEWORK - page 168 #1-16, 22-23

Application: Harmonic Oscillator

Newton's Law of Motion:

$$F = ma = m y''$$

Hooke's Law: The force exerted by a spring is proportional to the distance the spring is stretched.

$$F = k y$$

Shock Absorber: The force exerted by a shock absorber (dashpot) is proportional to its velocity.

$$F = c y'$$

Example…Automobile Suspension:
Assume a wheel on an automobile is suspended and the equilibrium position is given the value $y = 0$. At the equilibrium, the spring is not moving and so

$$mg = k\lambda,$$

where λ is the distance gravity pulls the spring from its natural length. The forces acting on the suspended suspension are:

force of gravity = mg pulling down
force in the spring = $k(y+\lambda)$
force in the shock absorber = cy'.

So,

$$my'' = mg - k(y+\lambda) - cy'$$

Hence, we have

$$my'' = k\lambda - k(y+\lambda) - cy' = ky - cy',$$

or

$$my'' + cy' + ky = 0.$$

If we also apply an external force $f(t)$ to the suspension (ie, we push on it), we obtain

$$my'' + cy' + ky = f(t),$$

where m = mass of object pulling string, k = spring constant, c = coefficient of damping.

Converting a second order DE to a first order system:

Consider a second order linear DE

$$y'' + p(t)y' + q(t)y = r(t).$$

Create the variable $v = y'$. Then,

$$y' = v$$
$$v' = r(t) - q(t)\, y - p(t)\, v$$

To find a particular solution, one generally has an initial condition which specifies $y(0)$ and $y'(0)$. (Initial location and velocity.)

HOMEWORK: page 182 #1-4, 8-11, 15

Euler's Method for Autonomous Systems: Consider the vector field. Starting at some point Y0 (IC) in this field, follow the direction specified by the corresponding vector F(Y0) for a distance t. Take this point as a new starting point and do it again. This generates an approximate solution curve for the system x' = f(x,y), y'=g(x,y) using the iteration:

$$x_{k+1} = x_k + t\, f(x_k, y_k)$$
$$y_{k+1} = y_k + t\, g(x_k, y_k)$$

Van der Pol Equation: page 188

Model for Swaying Skyscraper: page 191

HOMEWORK: page 194 #1-4, 5, 7,

Defn: For the autonomous system x'=f(x,y) and y'=g(x,y), the *x-nullcline* is the set of points (x,y) where f(x,y) is zero. The *y-nullcline* is the set of points (x,y) where g(x,y) is zero.

Note: On the x-nullcline, the vector field will be vertical. For the y-nullcline, the vector field will be horizontal. The points where the nullclines intersect will be the equilibrium points. We can use the regions that the nullclines break the x-y plane into and analyze the solution curves by noting the direction of travel along each nullcline (like the phase line).

HOMEWORK: page 207 #1-3, 7, 10-13

Skip section 2.7 on Lorenz systems