MATH 352 – Chapter 3
First Order Linear Systems

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Linear Systems of Equations with Constant Coefficients: **(SLC)**

Consider

$x' = ax + by$
$y' = cx + dy$

This is called a *homogeneous linear system of two equations with constant coefficients*.

Using matrix notation, we can rewrite as $\mathbf{Y}'(t) = \mathbf{A}\,\mathbf{Y}(t)$.

The system

$x' = ax + by + f(t)$
$y' = cx + dy + g(t)$

is *non-homogeneous* and can be written $\mathbf{Y}'(t) = \mathbf{A}\,\mathbf{Y}(t) + \mathbf{F}(t)$.

In general, an n-dimensional linear system of equations is of the form

$y_1' = a_{11}y_1 + a_{12}y_2 + ... + a_{1n}y_n$
$y_2' = a_{21}y_1 + a_{22}y_2 + ... + a_{2n}y_n$
$y_3' = a_{31}y_1 + a_{32}y_2 + ... + a_{3n}y_n$
. . .
$y_n' = a_{n1}y_1 + a_{n2}y_2 + ... + a_{nn}y_n$

Letting

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ \ldots & \ldots & & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

the above system can be written in the matrix form

$$\mathbf{Y}'(t) = \mathbf{A}\,\mathbf{Y}(t)$$

and will be denoted **(SLHC)**.

The non-homogeneous case

$$\mathbf{Y}'(t) = \mathbf{A}\,\mathbf{Y}(t) + \mathbf{F}(t),$$

will be denoted **(SLNC)**.

Converting LHC – When given an n^{th} order linear differential equations, one may convert to an n-dimensional first-order system by creating the temporary variables $y_1, y_2, ... , y_n$ using

$y_1 = y,$
$y_2 = y',$
$y_3 = y'',$
...
$y_n = y^{(n-1)}.$

Notice,

$y_1' = y_2$,
$y_2' = y_3$,
...
$y_{n-1}' = y_n$, and
$y_n' = -(a_{n-1}/a_n)\, y^{(n-1)} - \ldots - (a_1/a_n)\, y' - (a_0/a_n)\, y = -(a_{n-1}/a_n)\, y_{n-1} - \ldots - (a_1/a_n)\, y_2 - (a_0/a_n)\, y_1$

This can be easily written in system form $\mathbf{Y}' = \mathbf{A}\,\mathbf{Y}$. Similarly, one may convert the non-homogeneous case to (LNC).

Qualitative techniques discussed for systems in general still apply to linear systems and can be employed.

Defn: The *determinant* of a 2x2 matrix A is given by $a_{11}a_{22} - a_{12}a_{21}$. The determinant of a higher order system can be determined by using *cofactor expansion*.

Theorem: If det(A) is nonzero, then the only solution of $\mathbf{A}\,\mathbf{Y} = \mathbf{0}$ is the trivial solution. Hence, if det(A) is nonzero, the only equilibrium of the linear system is the origin. Conversely, a necessary condition for nontrivial equilibria is if det(A)=0.

Defn: If det(A)=0, then the matrix A is called *singular* or *degenerate*.

Superposition Principle (Linearity Principle): Consider the linear system $\mathbf{Y}'(t) = \mathbf{A}\,\mathbf{Y}(t)$.

- If $\mathbf{Y}(t)$ is a solution, then $k\mathbf{Y}(t)$ is also a solution
- If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, then $\mathbf{Y}_3(t) = \mathbf{Y}_1(t) + \mathbf{Y}_2(t)$ is also a solution.

The superposition principle implies that if we have two different solutions of (SLHC), then a two-parameter solution also exists. In general, for a n-dimensional system, we want to find a n-parameter solution involving linearly independent functions - the most general solution.

Defn: An *eigenvalue* of the matrix A is the scalar λ such that

$$\det(\lambda\mathbf{I} - \mathbf{A})=0.$$

For a given eigenvalue, the nontrivial vector $\mathbf{V}$ which satisfies $\mathbf{AV} = \lambda\mathbf{V}$ is called an *eigenvector*. Expanding $\det(\lambda\mathbf{I} - \mathbf{A})$ yields a polynomial expression in λ called the *characteristic polynomial*. Setting $\det(\lambda\mathbf{I} - \mathbf{A})=0$ yields the *characteristic equation*.

Suppose λ is an eigenvalue for the matrix **A** from (SLHC).
Consider the vector function $\mathbf{Y}(t) = e^{\lambda t}\,\mathbf{K}$, where K is some vector.
Then

$$\mathbf{AY} = \mathbf{Y}' = \lambda\, e^{\lambda t}\,\mathbf{K} = \lambda\mathbf{Y},$$

and so we get the expression $\mathbf{AY} = \lambda\mathbf{Y}$. Therefore, λ is an eigenvalue if and only if $\mathbf{Y}(t) = e^{\lambda t}\,\mathbf{K}$ is a solution of (SLHC).

Straight-Line Solutions - These solutions will satisfy the property that

$$\mathbf{AY} = \lambda\mathbf{Y},$$

for some scalar λ. Rewriting yields $\lambda\mathbf{Y} - \mathbf{AY} = \mathbf{0}$ or $(\lambda\mathbf{I} - \mathbf{A})\mathbf{Y} = 0$, which is true only if $\mathbf{Y}=\mathbf{0}$ or $\det(\lambda\mathbf{I} - \mathbf{A})=0$.

Remark: For straight-line solutions, positive eigenvalues correspond to solutions which travel away from the origin as time increases. Negative eigenvalues correspond to solutions which approach the origin as time increases.

Classifying equilibria for (SLHC) with distinct, real eigenvalues:
Saddle: (*Unstable*) A 2-dimensional linear system which includes a positive and a negative eigenvalue. Notice, solutions will tend to move past the origin and then asymptotic to the straight-line solutions.

Sink: (*Stable*) A 2-dimensional linear system which includes two negative eigenvalues. Note, solutions (except for one of the straight-line solutions) will tend toward the origin as t increases asymptotic to the straight-line solution corresponding the larger eigenvalue. Indeed, consider dy/dx = (dy/dt)/(dx/dt)...

Source: (*Unstable*) A 2-dimensional linear system which includes two positive eigenvalues. Note, solutions (except for one of the straight-line solutions) will tend toward the origin as t decreases asymptotic to the straight-line solution corresponding the smaller eigenvalue.

Application to the Harmonic Oscillator

Complex Eigenvalues: By the previous work, if λ is an eigenvalue with eigenvector **V**, $\mathbf{Y}(t) = e^{\lambda t}\,\mathbf{V}$ is still a solution. However, if $\lambda = a+bi$, then by using Euler's formula

$$e^{ib} = \cos(b) + i \sin(b)$$

and so we can obtain the solution

$$\mathbf{Y}(t) = e^{at}\{\cos(bt) + i \sin(bt)\}\mathbf{Y}_{\lambda} = \mathbf{Y}_{re}(t) + i\,\mathbf{Y}_{im}(t).$$

Result: For (SLHC) with a complex eigenvalue pair, $\mathbf{Y}_{re}(t)$ and $\mathbf{Y}_{im}(t)$ are real and linearly independent solutions. (Indeed, plug in and equate real parts and imaginary parts.)

Classifying equilibria for (SLHC) with distinct, complex eigenvalues: Consider the eigenvalue $\lambda = a + b\,i$.
Spiral Sink: If $a < 0$.

Spiral Source: If $a>0$

Center: If $a=0$

Natural Period: $2\pi/b$

Natural Frequency: $b/2\pi$

Zero Eigenvalues: Arise from degenerate systems. Suppose $\lambda=0$ is an eigenvalue with eigenvector **K**. Then,

$$\mathbf{Y_1}(t) = c_1\, e^{\lambda t}\,\mathbf{K} = c_1\,\mathbf{K}$$

is a constant solution, all of which are equilibrium points. Hence, solutions will tend to be attracted or repelled from this line of equilibrium points dependent upon whether the other eigenvalue is positive or negative.

Consider the function

$$\mathbf{Y_2}(t) = t\, e^{\lambda t}\,\mathbf{K} + e^{\lambda t}\,\mathbf{V},$$

where **V** is some other vector. Plugging this into the differential equation yields the requirement that

$$\mathbf{V} + \lambda\,\mathbf{K} = \mathbf{A}\,\mathbf{K},$$

or by rewriting $(\mathbf{A}-\lambda\mathbf{I})\mathbf{V} = \mathbf{K}$.

Repeated Eigenvalues: Suppose λ is an eigenvalue with eigenvector **V**. Then, $\mathbf{Y_1}(t) = e^{\lambda t}\,\mathbf{K}$, is known to be a solution. Consider the function $\mathbf{Y_2}(t) = t\, e^{\lambda t}\,\mathbf{V}$, where **V** is some other vector. Plugging this into the differential equation yields the requirement that $\mathbf{K} + \lambda\,\mathbf{V} = \mathbf{A}\,\mathbf{V}$, or by rewriting $(\mathbf{A}-\lambda\mathbf{I})\mathbf{V} = \mathbf{K}$.

Bifurcations: What happens if we let one term of the matrix remain variable

Linear Algebra...
To solve (SLHC):
Determine the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and corresponding eigenvectors $V_1, V_2, ..., V_n$.

Then, note $A\, V_k = \lambda_k\, V_k$, for $k = 1, 2, ... n$.

Create matrices $V = [V_1, V_2, \ldots, V_n]$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ = matrix with only the eigenvalues on the diagonal and zero elsewhere.

Then, it is easy to check that $A V = V \Lambda$.

So. $A = V \Lambda V^{-1}$ allows us to rewrite the original differential equation in the form

$Y' = V \Lambda V^{-1} Y$, or

$V^{-1} Y' = \Lambda V^{-1} Y$.

By setting $X = V^{-1} Y$ yields
$X' = \Lambda X$
which is a completely decoupled system of equations. This system easily has solution

$x_k(t) = c_k \exp(\lambda_k t)$, for $k = 1, 2, \ldots, n$.

So, the solution Y to the original system can be found via

$$\begin{aligned} Y &= VX \\ &= [V_1, V_2, \ldots, V_n] X \\ &= c_1 \exp(\lambda_1 t) V_1 + c_2 \exp(\lambda_2 t) V_2 + \ldots + c_n \exp(\lambda_n t) V_n, \end{aligned}$$

which is the same general solution as before.

Miscellaneous Other Topics of Interest using Linear Differential Equations

Newton's Law of Cooling -
Suppose the average rate of temperature change of a body is proportional to the difference in the temperature T of the outside medium and the temparature u(t) of the body itself. Then, $[u(t+\Delta t) - u(t)]/\Delta t = k[u(t) - T]$, or by taking limits gives $u'=k(u-T)$.

Momentum in a Gravitational Field -
The rate of change in momentum encountered by a moving object is directly proportional to the net force F applied to it. So, if m=mass and v(t)=velocity, momentum=mv gives $(mv)'=F$ or $vm'+mv'=F$. Since mass is assumed to be constant, then $m'=0$ gives $F = mv' = ma$. (Newton's Second Law)

Falling Bodies -
Newton's Law of gravitational attraction between two masses of size m_1 and m_2 states the force F between the two bodies is proportional to m_1m_2/r^2, where r is the distance between the two objects. That is, $F=Gm_1m_2/r^2$, where G is some constant. We will consider falling bodies near the earths surface. Hence, m_2=mass of earth>>m_1=mass of object and r is essentially constant. So, the gravitational force $F = (Gm_2/r^2)\, m_1 = g\, m_1$, where g is essentially a constant. So, as an object falls toward the earth, we get the following cases:
Case 1: W/o friction or external force: By (3), $F=m_1v'$. By above, $F=m_1g$.
By conservation of energy, we must have $m_1v' = m_1g$, or $a = v' = g$. Solving yields $v(t) = gt + C$.

Case 2: With frictional force but without an external force:
As above, except gravity is opposed by a frictional force proportional to the velocity of the object.
By conservation of energy, we must have $m_1v' = m_1g - kv$.

Case 3: With friction and a buoyant force:
As above, except from Archimedes' principle, gravity is also opposed by a force equal to the weight m_fg of the fluid displaced by the body. Then, $m_1v' = m_1g - kv - m_fg = (m_1-m_f)g - kv$. Notice what happens when m_1 is >, = and < m_f.

VARIATION OF PARAMETERS:
- The method of undetermined coefficients is not so hard to apply once you decide what your guess for y_p should be. However, if you can not determine an appropriate guess, it does not work. The method of variation of parameters is more robust in that it applys in all situations. The cost for this is extra effort in needing to do some integrations which may be hard or impossible to carry out.

Remark: Variation of Parameters is a powerful tool. We will only consider its use in solving (LNC). Skip section 4.3.
Remark: Variation of Parameters will work for nth order (LNC). We will derive the equations for n=2 only.

Derivation when n=2:
Let y_1 and y_2 solve (LH). Set $y = u\, y_1 + v\, y_2$, where u and v are unknown functions (parameters which can vary.)
Then $y' = (u\, y_1 + v\, y_2)' = u'y_1 + u\, y_1' + v'y_2 + v\, y_2'$ and $y'' = u''y_1 + u'y_1' + u'y_1' + u\, y_1'' + v''y_2 + v'y_2' + v'y_2' + v\, y_2''$.
Plugging into $y'' + p(x)y' + q(x)y = g(x)$ gives

$$(u''y_1 + u'y_1' + u'y_1' + u\, y_1'' + v''y_2 + v'y_2' + v'y_2' + v\, y_2'') + p(x)(u'y_1 + u\, y_1' + v'y_2 + v\, y_2') + q(x)(u\, y_1 + v\, y_2) = g(x), \text{ or}$$
$$u(y_1'' + p(x)y_1' + q(x)y_1) + v(y_2'' + p(x)y_2' + q(x)y_2) + 2(u'y_1' + v'y_2') + p(x)(u'y_1 + v'y_2) + u''y_1 + v''y_2 = g(x), \text{ or}$$
$$2(u'y_1' + v'y_2') + p(x)(u'y_1 + v'y_2) + u''y_1 + v''y_2 = g(x).$$

We have free choice of u and v. So, we want to choose them so that $u'y_1 + v'y_2 = 0$. Then also $0=(u'y_1 + v'y_2)'$ gives the original equation becomes finally $u'y_1' + v'y_2' = g(x)$. Therefore, if y_1 and y_2 solve (LH), then determining functions u and v such that

$$u'y_1 + v'y_2 = 0$$
$$u'y_1' + v'y_2' = g(x)$$

gives $y = u\, y_1 + v\, y_2$ solves (LN) as desired. Notice, the Wronskian of this system is nonzero provided y_1 and y_2 are linearly independent, which is assumed.

Actual Implementation: To solve (LN) in the case n=2:
1. Write the 2nd order (LN) problem in the form $y'' + p(x)y' + q(x)y = g(x)$.
2. Determine two linearly independent solutions y_1 and y_2 for (LH).
3. Solve the set of equations below for u' and v'.
   $u'y_1 + v'y_2 = 0$
   $u'y_1' + v'y_2' = g(x)$

4. Integrate u' to obtain u and v' to obtain v.
5. Form the general solution $y(x) = (c+u(x))y_1(x) + (d+v(x))y_2(x)$.

General Case: Wronskian

Application: Automobile Suspension System Revisited Again
- We now can solve the automobile suspension problem as the car is driving down the road subjected to bumps and other external forces. This was Model 4 from before $my'' + cy' + ky = f(t)$. Various choices for f(t) are illustrated on page 259.
- We often will rewrite the DE in the form $y'' + (c/m)y' + (k/m)y = f(t)$, or $y'' + (2d)y' + w^2y = f(t)$.

Remark: One can express the solutions to the above system graphically by plotting (t,x(t)) and (t,y(t)). However, often one uses only one graph parametrically by plotting (x(t), y(t)). Such a solution curve is called a *trajectory*, *path* or *orbit* and the xy-plane containing the trajectory is called the *phase plane* of the system.

Remark: For autonomous systems, the slope of the solution at any time is dependent only upon the solution values x and y and not upon time t. That is, not dependent upon when the solution arrives at the point (x,y). So, $dy/dx = (dy/dt) / (dx/dt)$ is uniquely determined at all values of time.

Results:
(1) There is at most one trajectory through any point in the phase plane.
(2) A trajectory that starts at a point other than a rest point cannot reach a rest point in a finite amount of time.
(3) No trajectory can cross itself unless it is a closed curve. If it is a closed curve, then the solution is periodic.
(4) A trajectory not starting at a rest point:
    (a) will move along the same trajectory regardless of starting time
    (b) cannot return to the starting point unless the solution is periodic.
    (c) can never cross another trajectory.
    (d) can only approach a rest point.

Remark: One may use Laplace transform methods for solving systems by taking the Laplace transform of all equations given. This yields now a system of equations in $X(s)=L\{x(t)\}$ and $Y(s)=L\{y(t)\}$. Solve these equations for X(s) and Y(s). Then apply inverse transforms.