MATH 352
Ordinary Differential Equations

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Chapter One

Mathematics is the study of quantity, structure, space, and change. Historically, mathematics developed from counting, calculation, measurement, and the study of the shapes and motions of physical objects, through the use of abstraction and deductive reasoning. (http://en.wikipedia.org/wiki/Mathematics).

Through a study of Differential Equations, the student will investigate, develop and utilize all of these facets of Mathematics. The goal is to apply the methods toward solving practical problems involving change.

Mathematical Models: Many processes in life can be analyzed using Mathematics. The process of representing reality mathematically is known as Mathematical Modeling. When working with models involving change, the independent variable generally is time and will be denoted by the variable t. The dependent variables will be functions of this independent variable and will often be denoted x(t) or y(t).

To create mathematical models is a difficult process and often one will make some simplifying assumptions in order to make the problem easier to formulate and solve. When doing so, one must be certain to interpret the "solution" as solving (perhaps exactly) an inexact model. Therefore, one must always take care when interpreting the "final answer" and some method of validation—checking to see if the answer conforms to reality—should always be employed.

Some common expressions:

"is" means "="

"A is proportional to B" means "A = kB", for k=constant

Ways to generate solutions for Mathematical Models:

- Analytic – Using explicit mathematical formulas to describe the solution .
- Qualitative - Using geometric techniques to describe the behavior of the solution.
- Numerical - Using approximation techniques to generate a discrete set of points which is "close" to the exact analytic solution.

Population growth model:
Let P(t) be the number of individuals in a particular population at time t. With unlimited resources, suppose the change in a population is affected only by the rate k of the difference in births and deaths per unit time. Then, over a change in time Δt,

$$P(t + \Delta t) - P(t) = (k \Delta t) P(t).$$

Dividing yields

$$\Delta P/\Delta t = k P(t).$$

Taking limits gives a *1st order Ordinary Differential Equation* (ODE):

$$P' = kP,$$
or
$$dP/dt = kP$$

Defn: A *Differential Equation* (DE) is an equation involving an unknown function and one or more of its derivatives.

- The *order* of the DE is the highest order derivative involved in the equation.
- A DE is *linear* if the unknown function occurs only to at most the first power.
- A *Solution* of a DE is a function which when substituted in satisfies the equation and perhaps any other supplied conditions.
- The *general solution* to an nth order DE is a solution (containing n arbitrary constants) that contains all possible solutions over a given interval.
- A *particular solution* is a general solution where the arbitrary constants have definite values.

Remark: We will start by considering first order DE's of the form y' = f(t,y) where the solution is the unknown function y(t).

Defn: When analyzing a given DE y' = f(t,y)

- *Equilibrium solution* - values where y' = 0, that is where there is no rate of change. An equilibrium will be *stable* if solutions near the equilibrium tend to approach the equilibrium as time progresses. If close solution tends to move away from the equilibrium, then the equilibrium is called *unstable*.
- *Initial condition* (IC) - y(0) is the value at time zero...generally assumed as a given parameter...y_0
- *Initial Value Problem* (IVP) - DE with an Initial Condition (IC): y' = f(t,y), y(0) = y_0
- *General solution* - a solution of the DE where parameters are left undetermined
- *Particular solution* - a solution of the DE where parameters are assigned appropriate values, often dependent upon IC.

<u>Ex</u>: For P'=kP above, determine:

- equilibrium solutions
- effects of IC
- general solution
- particular solutions

Logistic Population Model:
Suppose, in the population model developed above, that the rate of growth slows down as the population increases. In particular, assume that there is some maximum sustainable population M at which point the growth rate should become zero. A possible model:

$$\Delta P = (k(M-P)\, \Delta t)\, P(t).$$

and by taking limits yields

$$P' = k(M-P)P$$

Ex: The Logistic Model has:

- an unstable equilibrium at P = 0
- a stable equilibrium at P = M.
- an analytical solution, using separation of variables and partial fractions.

Note: We can rewrite the model algebraically as (P/M)' = Mk(1 - (P/M)) (P/M). Setting x = P/M and c = Mk yields the equivalent model x' = c(1-x)x. Many texts utilize this form for the logistic model.

HOMEWORK - page 14, #2, 6, 9, 20

Finding Analytic Solutions:

- If $x' = g(t)$, then DE can be solved by *simply integrating* $dx = g(t)\,dt$
- If $x' = g(t)h(x)$, then DE is *separable* and can be solved via $dx / h(x) = g(t)dt$
- If $x' = h(x)$, then DE is *autonomous* and can be solved via $dx / h(x) = dt$

Checking solutions: Given a function $x(t)$, one may check to see whether it is a solution by evaluating the given equation using this function and it's derivatives.

Note: The ODE $x' = 0$ is easily solved by integrating to obtain $y(t) = C$, where C is any constant. So, one ODE gave rise to a *family* of solutions, each one different yet only by a constant. Look at the graphical solution.

Ex: The solution to the ODE $x'' = 6t + 30$ may be found by simply integrating once to get $x' = 3t^2 + 30t + C$, and then integrating again to get the two-parameter family of solutions

$$x(t) = t^3 + 15t^2 + C\,t + D,$$

where C and D are any constants. To check the solution, take two derivatives to obtain the original DE.

Note: For the given *solution* $x(t) = C\,e^{-2t} + D\,e^{3t}$, we can eliminate the arbitrary constants by differentiating and combining the derivatives to get the differential equation $x'' - x' - 6x = 0$. Conversely, we can expect that in starting with a DE the solution will contain arbitrary constants.

Particular Solutions: When a DE is supplied with an initial condition, one obtains the particular solution by first solving for the general solution $x(t)$ involving all of its arbitrary constants. Then, by substituting the IC into this general solution, one may algebraically solve for the unknown constant(s).

Implicit Solutions: When employing various mathematical techniques for obtaining an analytic solution to a differential equation, one creates an equation involving the unknown solution $x(t)$ which cannot be solved explicitly for x in terms of the independent variable t. In this case, we will have to be satisfied with an (unfunctional) formulation for the solution and will be able to determine solution values using root-finding techniques. Indeed, when evaluating at a particular value of t, an equation will need to be solved each time for the unknown variable x.

HOMEWORK - page 33, #5-34 (as many as possible), #37

Group Homework – page 37, #40. Work this problem analytically. Attempt to create a similar mixing problem using something like strong (or weak) coffee/tea and plain water. Test your analytic solution to the results of your experiment.

Defn: Given the ODE y'=g(x,y), an isocline is the set of points (x,y) which satisfy g(x,y) = c, for c a given constant. That is, an isocline is a locus of point along which all solution curves which intersect this set have the same fixed slope. Graphing short "tangent vectors" along numerous isoclines gives the slope field or direction field of the solution. (Note, this involves selecting several 'slopes' c and solving for data points (x,y)...) This idea was discussed in Calculus IV when looking at "direction fields".

Note: The use of graphical differential equations software is very useful in doing direction fields for complicated functions since the amount of numerical calculation and then graphical rendering is quite tedious if not downright impossible.

Special Cases:

- y' = f(t) implies isoclines are vertical
- y' = f(y) implies isoclines are horizontal (the autonomous case)

Example: consider problem #16, page 51

HOMEWORK: page 48, work several. Download winplot.exe from http://math.exeter.edu/rparris/default.html . Use this software to develop slope fields for several of the given differential equations. Develop a short report using the software to completely describe the nature of the DE given in problem #12 on page 49.

Remark: We will sometimes use continuous equations and techniques to model problems which are necessarily discrete in nature (eg. population, etc.). Generally, this leads to reasonable models and solutions but not always. This also allows one to use the tools of calculus in determining solutions for the models. However, discrete analogs of DE's called *difference equations* may more precisely model the situation at hand. Further, these may be more readily applicable to computer solutions.

Numerical Techniques:
Given an IVP, a *numerical solution* is a collection of points (t_k, w_k) where $w_k \approx y(t_k)$.

Euler's method: Use definition of derivative and drop the limit with $h = t_{k+1} - t_k$. This suggests the iteration:

$$t_k = h\,k + t_0$$

$$w_0 = y_0$$
$$w_{k+1} = w_k + h\,f(t_k, w_k)$$

Ex: Apply to $P' = kP$ yields

$$w_0 = y_0$$
$$w_{k+1} = w_k + h\,k\,w_k = (1 + h\,k)w_k$$

General Numerical Solution is discrete points

Alternate derivations/interpretations of Euler's Method
- derivation using shooting in the slope field
- derivation using Taylor's series

Note: Numerical methods sometimes fail and we must apply them with care.

HOMEWORK: page 63, #2, 6, 12

Example: Consider the DE

$$y' = \frac{t}{yt^2 + y}$$

Notice the derivative is undefined when y = 0. However, for t < 0, y(t) approaches y = 0. What happens to the trajectory y(t)?

Existence and Uniqueness of Solutions:

- Existence Theorem - page 66 - Guarantees a solution to your IVP provided f(t,y) has continuous partial derivatives in a neighborhood of the IC.
- Uniqueness Theorem - page 68 - Guarantees that if you have found a solution satisfying IC, it is the only one.

Application of uniqueness result

- using equilibrium solutions - page 70
- using a known solutions - page 71
- to show that solutions to some ODEs can be asymptotic only to equilibrium solutions - page 71
- to check the validity of numerical solutions - page 72

HOMEWORK - page 73 #2, 5-8, 12, 15

Remark: Not all solutions y(t) to a given DE exist for all time even when the slope field is defined for all (t,y).

Ex: $y' = (1+y)^2$

Remark: When the DE is autonomous, we often consider the direction field just around the y-axis and call this the phase line.

Remark: Rest points can be classified according to how solution curves act near these points. As the independent variable tends to get large, if the solution curves stay close the equilibrium the rest point is said to be stable. Else, it is called unstable.

Classification of equilibrium solutions:

- Stable = sink
- Unstable = source
- Neither = node

Linearization Theorem - to help classify type of equilibrium point, consider the tangent line as an approximation to the slope function near the equilibrium point. (Taylor's polynomial of degree 1). Suppose y_0 is an equilibrium point of the autonomous DE y' = f(y). Then,

- if $f'(y_0)<0$, then y_0 is a sink;
- if $f'(y_0)>0$, then y_0 is a source;
- if $f'(y_0)=0$ or does not exist, then additional information is needed.

Pf: Let y(t) solve y' = f(y)

- If $f'(y_0)<0$, then f is decreasing at y_0.
  However, $f(y_0)=0$ implies $y'<0$ above y_0 and $y'>0$ below y_0.
  Therefore, y_0 is a sink.
- If $f'(y_0)>0$, then f is increasing at y_0.
  However, $f(y_0)=0$ implies $y'>0$ above y_0 and $y'<0$ below y_0.
  Therefore, y_0 is a source.

Look at different graphs for the third case to see that all types can occur.

Modified Logistic Model: To allow for both a maximum and minimum sustainable population

$$P' = k P (1 - P/M) (P/m – 1),$$

where M = maximum pop/carrying capacity, m = minimum sustainable pop/sparsity constant.

HOMEWORK: page 88, #4, 12, 14, 23-30, 34

BIFURCATIONS - When parameters are changed in a DE, one would expect the solutions to also change. If a small change in a parameter leads to a drastic change in the solution, we say a *bifurcation* has occurred.

For an autonomous DE, a *bifurcation value* of the parameter is a value for which the structure of the phase lines change. Collecting the phase lines for various values of the parameter give a *bifurcation diagram*.

Bifurcation Result: Consider the autonomous DE $y'=f_a(y)$, where the partials of $f_a(y)$ are continuous. For parameter $a = a_0$, consider the equilibrium $y=y_0$.

- Source: If $f'_{a0}(y_0)>0$, then the DE $y'=f_a(y)$ has a source for all a near a_0. Hence, no bifurcation.
- Sink: If $f'_{a0}(y_0)<0$, then the DE $y'=f_a(y)$ has a sink for all a near a_0. Hence, no bifurcation.

Bifurcation: If the family of solutions has a bifurcation, then $f_{a0}(y_0)=0$ and $f'_{a0}(y_0)=0$, for some $a = a_0$ and $y=y_0$.

Logistic Population Model with Harvesting:

$$P' = k P (1-P/M) - A,$$

where A is Harvested from the Population each time period. Let A be the parameter of interest, with $A \geq 0$.
So, our model can be written $P' = f_A(P)$

- Case 1: A=0...leads to standard Logistic Model (no Harvesting) and solutions approach P=M.
- Case 2: A>0...note $f_A(P)$ is a concave downward quadratic which shifts downward as A increases.

Solving $f_A(P)=0$ yields:

- For $0 < A \leq k M/4$, the sink continues to exist but at a smaller location.
- For $A > k M/4$, the sink ceases to exist and the Population moves toward extinction....that is, we harvested too much.

Logistic Population Model with minimum sustainable population and harvesting:

$$P' = k P (M - P) (P - m) - C$$

Questions:

- What happens as m increases toward M, with no harvesting (C = 0)
- What happens as C increases with m small relative to M.
- Notice, a relatively small harvest rate C can lead to extinction of a species under these conditions.

HOMEWORK: page 106 # 2, 8, 9, 16

LINEAR DIFFERENTIAL EQUATIONS:

Consider the following problem: Snow is falling at a rate that increases with time, but melts at a rate proportional to the depth of snow. Then, if S is the depth of the snow in inches at time t (in hours),

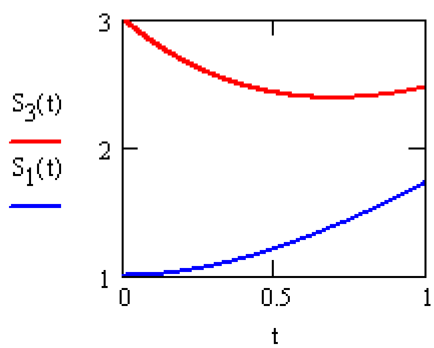
$$\Delta S = -a\Delta t S + r(t)\Delta t$$

Where a is constant and r(t) is the rate of snowfall at time t. The continuous DE model is given by

$$\tfrac{dS}{dt} = -aS + r(t).$$

For example, assuming that the snow falls at an increasing rate of r(t) = 1 + 2t and that the depth of the snow is 3 inches at time t=0. The depth of the snow as a function of time over a period of one hour behaves like the red line below. With an initial depth of 1 inch, the depth behaves like the blue line below.

$$\tfrac{dS}{dt} = -S + 1 + 2t$$



For S_3, the decrease in depth shows that in one hour more snow has melted than has accumulated. For S_1, the snow level continues to climb.

Additional question: Is there an initial depth c for which S(0) = S(1) (that is, the snowfall rate and the melting rate roughly balance each other in one hour)? We can determine that the general solution for the linear DE above is given by

$$S(t,c) := 2\cdot t - 1 + (c+1)\cdot e^{-t}$$

Solving for the initial depth c, evaluating S(0,c) = S(1,c) and solving yields

$$c := \frac{-(1+\exp(-1))}{(-1+\exp(-1))}$$, or

$$c = 2.164$$

Defn: A *first order linear* DE if of the form

y' = f(t,y) = g(t)y + r(t),

where g(t) and r(t) are continuous.

General solution: Rewrite the DE in the form...y' - g(t)y = r(t).
Let u(t) be an unknown function called an *integrating factor*. Note, D_t\{u y\} = u'y + y'u. Multiplying the DE by u(t) yields:

u(t) y' - u(t)g(t) y = u(t) r(t).

So the LHS fits the product rule if we can choose u so that

$$u' = -u\, g(t),$$

which is separable. If we choose u to be the solution of this separable DE, then the DE becomes

$$D_t\{u\, y\} = u(t)r(t),$$

which can be solved by integrating both sides with respect to t and solving for y(t).

HOMEWORK: page 115 #1-14, 22

Group ASSIGNMENT: LAB at the end of chapter 1